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Ordered Models for the Lambda Calculus.

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ORDERED MODELS FOR THE LAMBDA CALCULUS

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ABSTRACT

Although the λ-calculus has been studied for over fifty years, the first models for the lambda calculus were constructed by Dana Scott in 1972. Scott's models were constructed using inverse limits of continuous lattices. We investigate this inverse limit construction in the context of up-complete posets with zero.

We show that the correspondence between an up-complete poset with zero P and its associated model for the λ-calculus defines a monofunctor between appropriate categories. We calculate the values in the model corresponding to several combinators in the lambda calculus.

Finally, we investigate certain submonoids of the monoid of combinators. We show that this monoid contains as submonoids the non-negative integers and a free monoid on infinitely many generators.

Chapter 1

The Lambda Calculus

In this chapter we give a brief introduction to the λ-calculus, define environment models for the λ-calculus, and give sufficient conditions for the existence of environment models.

§ 1. Basic Definitions

Let V be a countable set (called the set of **variables**) which does not contain three symbols λ, (, and). Then the set Λ of λ-**terms** is the smallest set of words over the alphabet $V \cup \{\lambda, (,)\}$ such that

(1) $V \subseteq \Lambda$,

(2) $A, B \in \Lambda$ implies $(AB) \in \Lambda$, and

(3) $x \in V$ and $A \in \Lambda$ imply $(\lambda x A) \in \Lambda$.

A λ-term of the form (AB) is called an **application**, and a λ-term of the form $(\lambda x A)$ is called an **abstraction**. To make λ-terms readable, we write $\lambda x_1 \ldots x_n.A$ for $(\lambda x_1(\lambda x_2(\ldots(\lambda x_n A)\ldots)))$, and we always omit outermost parentheses. We also allow λ-terms to associate to the left. For example, we write ABC for $((AB)C)$. Variables occuring in λ-terms are **free** or **bound** according to the usual rules of logic, where we think of λ as a quantifier. Thus in $\lambda x.xy$, x is bound and y is free. We write $A \equiv B$ to indicate that A and B are the same λ-term. We reserve $A = B$ for another purpose, as we shall soon see. We identify all λ-terms which differ only by a renaming of bound variables. For example, $\lambda xz.xyz \equiv \lambda uv.uyv$.

Let $A, B \in \Lambda$. Then $A[x \leftarrow B]$ is the result of substituting B for each free occurrence of x in A. Some care is needed here (just as in the predicate calculus). If y occurs free in B, then the substitution of B into A must not cause y to become bound. If it is necessary, we rename bound occurrences of y in A. For example,

$$\lambda y.xy[x \leftarrow \lambda z.zy] \equiv \lambda u.xu[x \leftarrow \lambda z.zy] \equiv \lambda u.(\lambda z.zy)u,$$

where y is renamed as u in $\lambda y.xy$ in order to avoid binding the occurrence of y in $\lambda z.zy$. But $\lambda y.xy[x \leftarrow \lambda z.zy] \not\equiv \lambda y.(\lambda z.zy)y$.

The λ-calculus is a theory of equations, and is axiomatized as follows:

(1) $(\lambda x.A)B = A[x \leftarrow B]$ (β-conversion)

(2) $A = A$

(3) $A = B \Longrightarrow B = A$

(4) $A = B, B = C \Longrightarrow A = C$

(5) $A = B \Longrightarrow AC = BC$

(6) $A = B \Longrightarrow CA = CB$

(7) $A = B \Longrightarrow \lambda x.A = \lambda x.B$

We also consider the following optional axiom:

(8) $\lambda x.Ax = A$ (η-conversion)

When Axiom (8) is included, our theory is called the **extensional** λ-calculus. The intended intuitive interpretation of λ-terms is as follows. We think of AB as the application of A to B, where A is a function and B is in the domain of A. We think of $\lambda x.A$ as the function $x \mapsto A$. Thus, for example, we think of $\lambda x.x$ as representing the identity function, and $\lambda x.y$ as a constant function. Unfortunately, our intuition begins to falter when we are confronted with a λ-term such as xx, because we must think of x as a function and, at the same time, as an element of its own domain. We return to this point shortly, and show how sense can be made of xx.

A **combinator** is a λ-term with no free variables, and Λ_0 is the set of combinators. We now consider three important examples.

(1) Let $\mathbf{I} \equiv \lambda x.x$. Let $A \in \Lambda$. Then

$$\begin{aligned} \mathbf{I}A &\equiv (\lambda x.x)A \\ &= x[x \leftarrow A] \qquad (\text{ by } \beta\text{-conversion }) \\ &\equiv A. \end{aligned}$$

Thus $\mathbf{I}$ indeed behaves like an identity function.

(2) Let $\mathbf{K} \equiv \lambda xy.x$. Let $A, B \in \Lambda$. Then

$$\begin{aligned} \mathbf{K}AB &\equiv (\lambda xy.x)AB \\ &= (\lambda y.x)[x \leftarrow A]B \qquad (\text{ by } \beta\text{-conversion }) \\ &\equiv (\lambda y.A)B \\ &= A[y \leftarrow B] \qquad (\text{ by } \beta\text{-conversion }) \\ &\equiv A. \end{aligned}$$

Note that in the third and fourth lines of the calculation, y is a variable which does not occur free in A, and therefore the last line follows.

(3) Let $\mathbf{Y} = \lambda y.(\lambda x.y(xx))(\lambda x.y(xx))$. Let $A \in \Lambda$. Then

$$\begin{aligned}
\mathbf{Y}A &\equiv (\lambda y.(\lambda x.y(xx))(\lambda x.y(xx)))A \\
&= ((\lambda x.y(xx))(\lambda x.y(xx)))[y \leftarrow A] \\
&\equiv (\lambda x.A(xx))(\lambda x.A(xx)) \\
&= (A(xx))[x \leftarrow \lambda x.A(xx)] \\
&\equiv A((\lambda x.A(xx))(\lambda x.A(xx))) \\
&= A(\mathbf{Y}A).
\end{aligned}$$

Thus $\mathbf{Y}A$ is a "fixed point" for A. Remarks similar to those following Example (2) apply here.

§ 2. Environment Models

In order to understand the application xx, we need a way of identifying x both as a function, and as an element of the domain of that function. We therefore make the following definition: A **functional domain** is a set D equipped with a subset $[\ D \to D\ ]$ of D^D, and a surjective function $\Phi: D \to [\ D \to D\ ]$. Then elements of D correspond to functions on D via Φ, and every function in $[\ D \to D\ ]$ is representable as an element of D. Since Φ is surjective there exists $\Psi: [\ D \to D\ ] \to D$ such that $\Phi\Psi$ is the identity map on $[\ D \to D\ ]$. There is now a natural binary operation on D defined by $(a, b) \mapsto \Phi(a)(b)$.

We interpret lambda terms by assigning them values in D. The intention is that if $A \mapsto [\![A]\!]$ is the map assigning values in D, then application should correspond to the operation described above. That is, for two λ-terms A and B, $[\![AB]\!]$ should be the same as $\Phi([\![A]\!])([\![B]\!])$. We will see that such a map exists. An **environment** is a map $\rho: V \to D$. Thus an environment assigns an element of D to each variable. Given an environment ρ, we recursively extend ρ to a map $A \mapsto [\![A]\!]\rho: \Lambda \to D$ as follows. Let $A \in \Lambda$. Then

(1) If $A = x \in V$, then $[\![A]\!]\rho = \rho(x)$.

(2) If $A = BC$, then $[\![A]\!]\rho = \Phi([\![B]\!]\rho)([\![C]\!]\rho)$

(3) If $A = \lambda x.B$, then $[\![A]\!]\rho$ is defined as follows. Let $\rho\left\{\frac{a}{x}\right\}$ be a new environment defined by

$$\rho\left\{\frac{a}{x}\right\}(y) = \begin{cases} \rho(y) & \text{if } y \neq x; \\ a & \text{if } y = x \end{cases}$$

for $y \in V$. Then $[\![A]\!]\rho = \Psi(f)$, where $f: D \to D$ is defined by $f(a) = [\![B]\!]\rho\left\{\frac{a}{x}\right\}$ for $a \in D$. The interpretation of abstractions defined by (3) is best understood by thinking of the correspondence between ρ and $\rho\left\{\frac{a}{x}\right\}$ as a substitution which takes place in D, rather than in **A**. Of course the problem with the definition of $[\![\ ]\!]\rho$ is that the function f defined in (3) may not be in $[\,D \to D\,]$, and hence not in the domain of Ψ. We therefore make the following definition: **An environment model** of the λ-calculus is a functional domain D such that, when values are assigned to λ-terms as in (1),(2), and (3) above, then the functions f defined in (3) are always in $[\,D \to D\,]$. D is an **extensional** environment model if $\Psi = \Phi^{-1}$.

An equation $A = B$ in the λ-calculus is **valid** in an environment model D if $[\![A]\!]\rho = [\![B]\!]\rho$ for every environment ρ. Our definition of a model is justified by the following facts:

Lemma 2.1: $[\![A]\!]\rho = [\![A]\!]\rho\left\{\frac{a}{x}\right\}$ for every $a \in D$ and x which is not free in A.

Thus the value of a λ-term depends only on the values assigned to its free variables by ρ. In particular, if A is a combinator, then $[\![A]\!]\rho$ is independent of the choice of ρ, and we may just write $[\![A]\!]$.

Lemma 2.2: $[\![A[x \leftarrow B]]\!]\rho = [\![A]\!]\rho\left\{\frac{[\![B]\!]\rho}{x}\right\}$. Thus substitution behaves well under evaluation in the model.

Theorem 2.3: If $A = B$ is provable in the λ-calculus, then $A = B$ is valid in D.

Theorem 2.4: If D is an extensional environment model, and $A = B$ is provable in the extensional λ-calculus, then $A = B$ is valid in D.

These results are all easily obtained using induction on the composition of a λ-term. They (and the definition of an environment model) are found in Meyer [1981].

In order to give an example which shows that environment models closely reflect the structure of the λ-calculus, we examine the behavior of $[\![\mathbf{Y}]\!]$ in an environment model D.

Proposition 2.5: For each $a \in D$, $\Phi([\![\mathbf{Y}]\!])(a)$ is a fixed point of $\Phi(a)$.

Proof: Let $a \in D$. Let ρ be an environment such that $\rho(x) = a$. Then

$$\begin{aligned}\Phi([\![\mathbf{Y}]\!]\rho)(a) &= \Phi([\![\mathbf{Y}]\!])(\rho(x)) \\ &= \Phi([\![\mathbf{Y}]\!]\rho)([\![x]\!]\rho) \\ &= [\![\mathbf{Y}x]\!]\rho \\ &= [\![x(\mathbf{Y}x)]\!]\rho \\ &= \Phi([\![x]\!]\rho)([\![\mathbf{Y}x]\!]\rho) \\ &= \Phi(a)(\Phi([\![\mathbf{Y}]\!]\rho)(a)).\end{aligned}$$

Thus $\Phi([\![\mathbf{Y}]\!])(a)$ is a fixed point of $\Phi(a)$. ♣

§ 3. The Existence of Environment Models

We now establish sufficient conditions for the existence of environment models. In Chapter 3 we actually construct extensional environment models.

Theorem 3.1: Let X be a set, and let C be a category whose objects are the Cartesian powers $X^n = \times_{i=1}^{n} X$ of X and whose morphisms are functions. Let C satisfy the following conditions:

(1) For each $n \geq 0$, $\mathbf{Mor}(X^n, X)$ contains all projections and constant maps.

(2) If $f, g \in \mathbf{Mor}(X^n, X)$, then $f \times g \in \mathbf{Mor}(X^n, X \times X)$, where $f \times g : X^n \to X \times X$ is defined by $f \times g(x_1, \ldots, x_n) = \big(f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)\big)$.

(3) For each $f \in \mathbf{Mor}(X^{n+1}, X)$, $c_n(f)(X^n) \subseteq \mathbf{Mor}(X, X)$, where

$$c_n : \mathbf{Mor}(X^{n+1}, X) \to (X^X)^{(X^n)}$$

is defined by $c_n(f)(x_1, \ldots, x_n)(x) = f(x_1, \ldots, x_n, x)$.

(4) There are maps $\Phi : X \to \mathbf{Mor}(X, X)$ and $\Psi : \mathbf{Mor}(X, X) \to X$ such that:

(a) $\Phi\Psi$ is the identity map on $\mathbf{Mor}(X, X)$,

(b) The map $\mathrm{App} : X \times X \to X$ defined by $\mathrm{App}(a, b) = \Phi(a)(b)$ is in $\mathbf{Mor}(X \times X, X)$, and

(c) for every $g \in \mathbf{Mor}(X^{n+1}, X)$, $\Psi \circ \big(c_n(g)\big) \in \mathbf{Mor}(X^n, X)$.

Then X is an environment model for the lambda calculus.

Proof: We must show that when λ-terms are assigned values as in the previous section, then all functions of the form $a \mapsto [\![A]\!]\rho\left\{ \frac{a}{x} \right\}$ are in $\mathbf{Mor}(X, X)$.

We will actually prove the following by induction on the structure of λ-terms: All functions from X^n to X of the form $(a_1,\ldots,a_n)\mapsto [A]\rho\left\{\frac{a_1}{x_1}\right\}\left\{\frac{a_2}{x_2}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}$, where $\{x_1\ldots x_n\}$ are distinct variables, are in $\mathbf{Mor}(X^n,X)$.

Let A be a variable, say x, which is distinct from $x_1,\ldots,x_n$. Then

$$f(a_1,\ldots,a_n)=[x]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}=\rho(x).$$

Thus f is a constant map, and therefore is in $\mathbf{Mor}(X^n,X)$.

Suppose that $A=x_i$. Then $f(a_1,\ldots,a_n)=[x_i]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}=a_i$. Thus f is a projection, and is therefore in $\mathbf{Mor}(X^n,X)$.

Now let A be a λ-term which is not a variable, and assume that all functions of the form $(a_1,\ldots,a_m)\mapsto [B]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_m}{x_m}\right\}$ are in $\mathbf{Mor}(X^m,X)$, for all λ-terms B of simpler structure than A, and for all m.

Let $A=BC$. Then

$$\begin{aligned}f(a_1,\ldots,a_n)&=[BC]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}\\&=\Phi\left([B]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}\right)\left([C]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}\right)\\&=\mathrm{App}\left([B]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\},[C]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}\right)\\&=\mathrm{App}\circ(g\times h)(a_1,\ldots,a_n),\end{aligned}$$

where $g\colon X^n\to X$ and $h\colon X^n\to X$ are defined by $g(a_1,\ldots,a_n)=[B]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}$ and $h(a_1,\ldots,a_n)=[BC]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}$. By the induction hypothesis, g and h are in $\mathbf{Mor}(X^n,X)$, and so $g\times h\in\mathbf{Mor}(X^n,X\times X)$. Thus $\mathrm{App}\circ(g\times h)\in\mathbf{Mor}(X^n,X\times X)$.

Let $A=\lambda x.B$, where A is a variable, say x, distinct from $x_1,\ldots,x_n$. Then

$$f(a_1,\ldots,a_n)=[\lambda x.B]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}=\Psi(g),$$

where $g\colon X^n\to X$ is defined by $g(b)=[B]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}\left\{\frac{b}{x}\right\}$. Define $\hat{g}\colon X^{n+1}\to X$ by $\hat{g}(a_1,\ldots,a_n,b)=g(b)$. By the induction hypothesis, $\hat{g}\in\mathbf{Mor}(X^{n+1})$. Now

$$c_n(\hat{g})(a_1,\ldots,a_n)(b)=\hat{g}(a_1,\ldots,a_n,b)=g(b)$$

for $b\in X$. Thus $g=c_n(\hat{g})(a_1,\ldots,a_n)$. Therefore $f(a_1,\ldots,a_n)=\Psi\Big(\big(c_n(\hat{g})\big)(a_1,\ldots,a_n)\Big)$. Thus $f=\Psi\circ\big(c_n(\hat{g})\big)\in\mathbf{Mor}(X^n,X)$.

Let $A=\lambda x_i.B$. Then

$$f(a_1,\ldots,a_n)=[\lambda x_i.B]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}=\Psi(g),$$

where $g: X \to X$ is defined by $g(b) = [B]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}\left\{\frac{b}{x_i}\right\}$. Define $\mathfrak{g}: X^{n+1} \to X$ by $\mathfrak{g}(a_1, \ldots, a_n, b) = g(b)$. Let x be a variable which is not in $\{x_1, \ldots, x_n\}$. Then

$$\mathfrak{g}(a_1, \ldots, a_n, b) = [B]\rho\left\{\frac{a_1}{x_1}\right\}\ldots\left\{\frac{a_{i-1}}{x_{i-1}}\right\}\left\{\frac{a_i}{x}\right\}\left\{\frac{a_{i+1}}{x_{i+1}}\right\}\ldots\left\{\frac{a_n}{x_n}\right\}\left\{\frac{b}{x_i}\right\}.$$

Thus $\mathfrak{g} \in \mathrm{Mor}(X^{n+1}, X)$ by the induction hypothesis. As before, $f = \Psi \circ \left(c_n(\mathfrak{g})\right) \in \mathrm{Mor}(X^n, X)$. $\clubsuit$

In Chapter 3 we actually construct non-trivial extensional environment models.

Chapter 2

Up-complete Posets with Zero

In this chapter we develop enough of the theory of up-complete posets with zero and the Scott topology to construct models for the lambda calculus. The original construction is due to Scott [1972] in the context of continuous lattices. The construction is given in Barendregt [1981] for up-complete posets with zero. However, we follow Gierz, et.al. [1980], in considering a more general construction which yields fixed points of functors.

§ 1. Posets

A **partially ordered set**, or **poset**, is a set P equipped with a partial order $\leq$. We will often write $y \geq x$ rather than $x \leq y$. Let S be a subset of P. Then an element x of P is an **upper bound** of S if $x \geq y$ for all $y \in S$. An element x of P is a **lower bound** of S if $x \leq y$ for all $y \in S$. An upper bound x of S such that $x \leq y$ for all upper bounds y of S is the **least upper bound** of S, or **supremum**, of S. A lower bound x of S such that $x \geq y$ for all lower bounds y of S is the **greatest lower bound** of S, or **infimum**, of S. Note that least upper bounds and greatest lower bounds are unique (when they exist). We write $\sup S$ for the least upper bound of S and $\inf S$ for the greatest lower bound of S. If $\sup S \in S$ then we will sometimes write $\max S$ for $\sup S$. Similarly, if $\inf S \in S$ then we will sometimes write $\min S$ for $\inf S$. If $S \subseteq P$, then we define $\uparrow S$ by

$$\uparrow S = \{x \mid x \geq y \text{ for some } y \in S\}.$$

Similarly,

$$\downarrow S = \{x \mid x \leq y \text{ for some } y \in S\}.$$

For $x \in P$, we abbreviate $\uparrow \{x\}$ by $\uparrow x$ and $\downarrow \{x\}$ by $\downarrow x$. An element 0 of P is the **zero** of P if $0 \leq x$ for every $x \in P$. (I.e., $0 = \min P$.)

Let P be a poset. Then a set $D \subseteq P$ is **directed** if D is nonempty and every finite subset of D has an upper bound in D. Note that D is directed if and only if D is nonempty, and for every pair x, y of elements of D there exists $z \in D$ such that $x \leq z$ and $y \leq z$. P is up-complete if for every directed set $D \subseteq P$, $\sup D$ exists (in P).

Let P and Q be posets, and $f: P \to Q$. Then f is **order-preserving** if for all $x, y \in P$, $x \leq y$ implies $f(x) \leq f(y)$. We say that f is an **order isomorphism** if f is a bijection and f^{-1} is order-preserving.

Lemma 1.1: Let P and Q be posets, and let $f: P \to Q$ be an order-preserving map. Then $f(D)$ is a directed subset of Q whenever D is a directed subset of P.

Proof: If D is nonempty, then clearly $f(D)$ is nonempty. Let $x, y \in f(D)$. Then $x = f(u)$ and $y = f(v)$, where $u, v \in D$. Choose $w \in D$ such that $u \leq w$ and $v \leq w$. Then $f(u) \leq f(w)$, $f(v) \leq f(w)$, and $f(w) \in f(D)$. Thus $f(D)$ is directed.

§ 2. The Scott Topology

We now define the well-known **Scott topology** on a poset P. A subset U of P is **Scott open** if the following conditions are satisfied:

(1) $U = \uparrow U$.

(2) For every directed set $D \subseteq P$, $\sup D \in U$ implies $D \cap U \neq \emptyset$.

Proposition 2.1: The Scott open subsets of a poset P form a topology on P.

Proof: Clearly $\emptyset$ and P are Scott open sets. Let $\{U_i\}_{i\in I}$ be a family of Scott open sets. We show that $\bigcup_{i\in I} U_i$ is Scott open. It is clear that $\bigcup_{i\in I} U_i \subseteq \uparrow \bigcup_{i\in I} U_i$. Let $x \in \uparrow \bigcup_{i\in I} U_i$. Then $x \geq y$ for some $y \in \bigcup_{i\in I} U_i$. Now $y \in U_j$ for some $j \in I$, and so $x \in \bigcup_{i\in I} U_i$. We have shown that $\uparrow \bigcup_{i\in I} U_i = \bigcup_{i\in I} U_i$. Let $D \subseteq P$ be a directed set with $\sup D \in \bigcup_{i\in I} U_i$. Then $\sup D \in U_j$ for some $j \in I$, and so $D \cap U_j \neq \emptyset$. Thus $D \cap (\bigcup_{i\in I} U_i) \neq \emptyset$. It follows that $\bigcup_{i\in I} U_i$ is Scott open.

Now let $\{U_i\}_{i\in I}$ be a finite family of Scott open sets. We will show that $\bigcap_{i\in I} U_i$ is Scott open. Let $x \in \uparrow \bigcap_{i\in I} U_i$. Then $x \geq y$ for some $y \in \bigcap_{i\in I} U_i$. Now $y \in U_i$ for all $i \in I$, and so $x \in U_i$ for all $i \in I$. Thus $x \in \bigcap_{i\in I} U_i$. Hence $\bigcap_{i\in I} U_i = \uparrow \bigcap_{i\in I} U_i$. Let $D \subseteq P$ be a directed set with $\sup D \in U_i$ for all $i \in I$. Then $\sup D \in U_i$ for all $i \in I$, and so $D \cap U_i \neq \emptyset$ for all $i \in I$. Choose $x_i \in D \cap U_i$ for each $i \in I$. Let x be an upper bound in D for $\{x_i\}_{i\in I}$. Then for each $i \in I$, $x \geq x_i$, so $x \in U_i$. Thus $x \in U_i$ for all $i \in I$, and so $x \in \bigcap_{i\in I} U_i$. Hence $D \cap (\bigcap_{i\in I} U_i) \neq \emptyset$. It follows that $\bigcap_{i\in I} U_i$ is Scott open. ♣

Let P be a poset. The topology on P consisting of Scott open sets is called the **Scott topology**. A subset A of P is **Scott closed** if it is closed with respect to the Scott topology.

Proposition 2.2: Let P be a poset, and $A \subseteq P$. Then A is Scott closed if and only if it satisfies the following conditions:

(1) $A = \downarrow A$

(2) If D is a directed subset of A such that $\sup D$ exists, then $\sup D \in A$.

Proof: Let A be Scott closed. Then $P \setminus A$ is Scott open. Thus $P \setminus A = \uparrow (P \setminus A)$. Let $x \in \downarrow A$. Then $x \leq y$ for some $y \in A$. If $x \in P \setminus A$, then $y \in \uparrow (P \setminus A) = P \setminus A$, an impossibility. Thus $x \in A$, and so $\downarrow A = A$. Let D be a directed set in P such that $D \subseteq A$. If $\sup D \in P \setminus A$, then $D \cap (P \setminus A) \neq \emptyset$, an impossibility. Hence $\sup D \in A$. Thus A satisfies (1) and (2).

Conversely, let A satisfy conditions (1) and (2). We will show that $P \setminus A$ is Scott open. Let $x \in \uparrow (P \setminus A)$. Then $x \geq y$ for some $y \in P \setminus A$. If $x \in A$, then $y \in \downarrow A$, and so $y \in A$, an impossibility. Hence $x \in P \setminus A$, and so $\uparrow (P \setminus A) = P \setminus A$. Let D be a directed set in P such that $\sup D \in P \setminus A$. If $D \bigcap (P \setminus A) = \emptyset$, then $D \subseteq A$ and so $\sup D \in A$, an impossibility. Thus $D \bigcap (P \setminus A) \neq \emptyset$. We have shown that $P \setminus A$ is Scott open, and so A is Scott closed. ♣

Proposition 2.3: Let P be a poset and $x \in P$. Then the closure of $\{x\}$ in the Scott topology is $\downarrow x$.

Proof: First we show that $\downarrow x$ is Scott closed. Clearly $\downarrow (\downarrow x) = \downarrow x$. Let D be a directed set in P with $D \subseteq \downarrow x$. Then x is an upper bound on D, and so $\sup D \leq x$. Thus $\sup D \in \downarrow x$. It follows that $\downarrow x$ is Scott closed.

Let A be a Scott closed set in P with $\{x\} \subseteq A$. Then $\downarrow x \subseteq A$ since $\downarrow A = A$. Thus $\downarrow x$ is the closure of $\{x\}$. ♣

Corollary 2.4: The Scott topology on a poset P is a T_0 topology.

Proof: Let x and y be distinct elements of a poset P. Then either $x \not\leq y$ or $y \not\leq x$. Suppose without loss of generality that $x \not\leq y$. Then $x \in P \setminus (\downarrow y)$, and $P \setminus (\downarrow y)$ is a Scott open set which misses y. ♣

Corollary 2.5: If the Scott topology on a poset P is T_1, then the partial order on P is the diagonal order. (I.e., for $x, y \in P$, $x \leq y$ if and only if $x = y$.)

Proof: Suppose the Scott topology on P is T_1. Then for $x \in P$, the closure of $\{x\}$ is $\{x\}$, so $\{x\} = \downarrow x$. Thus $y \leq x$ if and only if $y = x$. ♣

§ 3. Scott continuous maps

Let P and Q be posets. A map $f: P \to Q$ is **Scott continuous** if f is continuous with respect to the Scott topology. In this section we will characterize Scott continuous maps in terms of their order properties.

Proposition 3.1: Scott continuous maps preserve order.

Proof: Let P and Q be posets, and let $f: P \to Q$ be Scott continuous. Let $x, y \in P$, with

$x \leq y$. Then $x \in \downarrow y = \overline{\{y\}}$. Thus

$$f(x) \in f\left(\overline{\{y\}}\right) \subseteq \overline{f(\{y\})} = \overline{f(y)} = \downarrow f(y),$$

and so $f(x) \leq f(y)$. Thus f is order-preserving. ♣

Corollary 3.2: Let P and Q be up-complete posets with zero. Let D be a directed subset of P, and $f: P \to Q$ be an order-preserving map. Then $f(D)$ is directed by Lemma 2.1.1, and so $\sup f(D)$ exists.

Lemma 3.3: Let P and Q be up-complete posets with zero, let $f: P \to Q$ be order-preserving, and let D be a directed subset of P. Then $f(\sup D)$ is an upper bound on $f(D)$, and hence $\sup f(D) \leq f(\sup D)$.

Proof: Let $d \in D$. Then $d \leq \sup D$, and so $f(d) \leq f(\sup D)$. Thus $f(\sup D)$ is an upper bound on $f(D)$. ♣

Proposition 3.4: Let P and Q be up-complete posets with zero, and $f: P \to Q$. Then f is Scott continuous if and only if $f(\sup D) = \sup f(D)$ for every directed subset D of P.

Proof: Suppose that f is Scott continuous. Then f is order-preserving, and so $f(\sup D)$ is an upper bound on $f(D)$ by Lemma 2.3.3. Let x be an upper bound on $f(D)$. Then $f(D) \subseteq \downarrow x$. Thus $\sup f(D) \in \downarrow x$ since $f(D)$ is directed and $\downarrow x$ is Scott closed. Thus $\sup f(D) \leq x$. We have shown that $\sup f(D) = f(\sup D)$.

Conversely, suppose that $f(\sup D) = \sup f(D)$ for every directed set $D \subseteq P$. Let V be a Scott open set in Q. We will show that $f^{-1}(V)$ is Scott open in P. Let $x \in f^{-1}(V)$, and let $y \geq x$. Then $f(x) \in V$, and $f(y) \geq f(x)$. Thus $f(y) \in V$, and so $y \in f^{-1}(V)$. It follows that $f^{-1}(V) = \uparrow f^{-1}(V)$. Let D be a directed set in P, with $\sup D \in f^{-1}(V)$. Then $\sup f(D) = f(\sup D) \in V$. Now $f(D) \bigcap V \neq \emptyset$ since $f(D)$ is directed. Let $d \in D$, with $f(d) \in V$. Then $d \in D \bigcap f^{-1}(V)$ and so $D \bigcap f^{-1}(V) \neq \emptyset$. ♣

Proposition 3.5: Let P and Q be up-complete posets with zero, and let $f: P \to Q$ be an order-isomorphism. Then f is a homeomorphism with respect to the Scott topology.

Proof: Let D be a directed set in P. Then $f(\sup D)$ is an upper bound on $f(D)$ by Lemma 2.3.3. Let y be an upper bound on $f(D)$. Then for $x \in D$, $f(x) \leq y$, and so $x = f^{-1}f(x) \leq f^{-1}(y)$. Thus $g(y)$ is an upper bound on D, and hence $\sup D \leq f^{-1}(y)$. Thus

$f(\sup D) \leq f f^{-1}(y) = y$. It follows that $f(\sup D) = \sup f(D)$, and so f is Scott continuous. Similarly f^{-1} is Scott continuous. ♣

§ 4. Products

Let $\{P_i\}_{i\in I}$ be a family of posets. The Cartesian product of the family is $\prod_{i\in I} P_i$, and $\pi_j : \prod_{i\in I} P_i \to P_j$ denotes the usual j^{th} projection. We order $\prod_{i\in I} P_i$ as follows: for $x, y \in \prod_{i\in I} P_i$,

$$x \leq y \text{ if and only if } \pi_j(x) \leq \pi_j(y) \text{ for all } j \in I.$$

It is clear that this is a partial order, and that the π_j are order-preserving.

Proposition 4.1: Let $\{P_i\}_{i\in I}$ be a family of up-complete posets with zero. Then $\prod_{i\in I} P_i$ is an up-complete poset with zero, and each $\pi_j : \prod_{i\in I} P_i \to P_j$ is Scott-continuous.

Proof: Let 0_j denote the zero of P_j for each $j \in I$. Define $0 \in \prod_{i\in I} P_i$ by $\pi_j(0) = 0_j$ for all $j \in I$. Then 0 is clearly a zero for $\prod_{i\in I} P_i$. Now $\pi_j(D)$ is directed for each $j \in I$ by Lemma 2.3.3. Thus we may define $d \in \prod_{i\in I} P_i$ by $\pi_j(d) = \sup \pi_j(D)$ for all $j \in I$. Then for $x \in D$, $\pi_j(x) \leq \pi_j(d)$ for all $j \in I$, so that $x \leq d$. Thus d is an upper bound of D. Let y be an upper bound on D. Then if $s \in \pi_j(D)$, $s = \pi_j(x)$ for some $x \in D$. Now $x \leq y$, and so $s \leq \pi_j(y)$. Thus $\pi_j(y)$ is an upper bound on $\pi_j(D)$, and so $\pi_j(d) \leq \pi_j(y)$. This is true for all $j \in I$, and so $d \leq y$. It follows that $d = \sup D$, and that $\prod_{i\in I} P_i$ is an up-complete poset with zero. Furthermore, we have shown that $\pi_j(\sup D) = \sup \pi_j(D)$, and hence that each π_j is Scott-continuous. ♣

Lemma 4.2: Let $\{P_i\}_{i\in I}$ be a family of up-complete posets with zero, and let $j \in I$. Let $x_i \in P_i$ for each $i \in I$ with $i \neq j$. Define $e_j : P_j \to \prod_{i\in I} P_i$ by

$$\pi_k e_j(x) = \begin{cases} x_k & \text{if } k \neq j \\ x & \text{if } k = j \end{cases}$$

for $k \in I$ and $x \in P_j$. Then e_j is Scott continuous.

Proof: Let $x, y \in P_j$, with $x \leq y$. Let $k \in I$. If $k \neq j$, then $\pi_k e_j(x) = x_k = \pi_k e_j(y)$. If $k = j$, then $\pi_k e_j(x) = x \leq y = \pi_k e_j(y)$. Therefore $\pi_j e_j(x) \leq \pi_k e_j(y)$ for all $k \in I$, and so $e_j(x) \leq e_j(y)$. Thus e_j is order-preserving.

Let D be a directed set in P_j. Then $e_j(\sup D)$ is an upper bound of $e_j(D)$ by Lemma 2.3.3. Let z be an upper bound of $e_j(D)$. Let $k \in I$. Assume that $k \neq j$. Then $\pi_k e_j(\sup D) = x_j$.

Let $d \in D$. Then $e_j(d) \leq x$, and so $x_k = \pi_k e_j(d) \leq \pi_k(x)$. Thus $\pi_k e_j(\sup D) \leq \pi_k(x)$. Now assume that $k = j$. Then $\pi_k e_j(\sup D) = \sup D$. For $d \in D, e_j(d) \leq x$, and therefore $d = \pi_k e_j(d) \leq \pi_k(x)$. Thus $\pi_k(x)$ is an upper bound of D, and so $\sup D \leq \pi_k(x)$. We have shown that $\pi_k e_j(\sup D) \leq \pi_k(x)$ for all $k \in I$, and it follows that $e_j(\sup D) = \sup e_j(D)$. Thus e_j is Scott continuous. ♣

Lemma 4.8: Let P, Q, and R be up-complete posets with zero. Let $f: P \to Q$ and $g: P \to R$ be Scott continuous maps. Define $f \times g: P \to Q \times R$ by $f \times g(x) = (f(x), g(x))$ for $x \in P$. Then $f \times g$ is Scott continuous.

Proof: Clearly $f \times g$ is order-preserving. Let D be a directed subset of P. Then $(f \times g)(\sup D)$ is an upper bound of $(f \times g)(D)$ by Lemma 2.3.3. Let y be an upper bound of $(f \times g)(D)$. Let $d \in D$. Then $(f \times g)(d) = (f(d), g(d)) \leq y$. Thus $f(d) \leq \pi_1(y)$, and $g(d) \leq \pi_2(y)$. Therefore $\pi_1(y)$ is an upper bound of $f(D)$, and $\pi_2(y)$ is an upper bound of $g(D)$. Thus $\sup f(D) \leq \pi_1(y)$, and $\sup g(D) \leq \pi_2(d)$. It follows that $(f \times g)(\sup D) = (f(\sup D), g(\sup D)) \leq y$. Thus $(f \times g)(\sup D) = \sup (f \times g)(D)$. ♣

§ 5. Adjoints

Let P and Q be posets. A pair (f, g) of order-preserving maps $f: P \to Q$ and $g: Q \to P$ is an adjunction between P and Q if $f(x) \geq y$ is equivalent to $x \geq g(y)$. We call f an **upper adjoint** and g a **lower adjoint**.

Proposition 5.1: Lower adjoints preserve sups.

Proof: Let (f, g) be an adjunction between P and Q. Let $S \subseteq Q$. Assume that $\sup S$ exists. Now $g(\sup S)$ is an upper bound of $g(S)$ by Lemma 2.3.3. Let x be an upper bound of $g(S)$. Then for $y \in S$, $x \geq g(y)$, so $f(x) \geq y$. Thus $f(x)$ is an upper bound of S, and so $f(x) \geq \sup S$. Hence $x \geq g(\sup S)$. It follows that $g(\sup S) = \sup g(S)$. ♣

Proposition 5.2: Let P and Q be posets, and let $f: P \to Q$ and $g: Q \to P$ be order-preserving maps. Let I_P be the identity map of P, and let I_Q be identity map on Q. Then the following are equivalent:

(1) (f, g) is an adjunction between P and Q, and f is surjective. (2) (f, g) is an adjunction between P and Q, and $g(y) = \min f^{-1}(y)$ for all $y \in Q$. (3) $fg = I_Q$, and $gf \leq I_P$.

Proof (1) $\Longrightarrow$ (2).

Assume that (1) holds. Let $x \in f^{-1}(y)$. Then $f(x) = y$, so $f(x) \geq y$. Thus $x \geq g(y)$, and so $g(y)$ is a lower bound on $f^{-1}(y)$. Since f is order-preserving, $f(x) \geq f(g(y))$. But $g(y) \geq g(y)$, so $f(g(y)) \geq y$. Thus $y = f(x) \geq f(g(y)) \geq y$, and so $y = f(g(y))$. Thus $g(y) \in f^{-1}(y)$. It follows that $g(y) = \min f^{-1}(y)$.

(2) $\Longrightarrow$ (3).

Assume that (2) holds. Let $y \in Q$. Then $g(y) \in f^{-1}(y)$, so $f(g(y)) = y$. Thus $fg = I_Q$. Let $x \in P$. Then $f(x) \geq f(x)$, so $x \geq g(f(x))$. Thus $gf \leq I_P$.

(3) $\Longrightarrow$ (1).

Assume that (3) holds. Clearly f is surjective. If $f(x) \geq y$, then $x \geq gf(x) \geq g(y)$ since g is order-preserving. If $x \geq g(y)$, then $f(x) \geq fg(y) = y$. Thus $f(x) \geq y$ if and only if $x \geq g(y)$. ♣

Note that (2) shows that an upper adjoint uniquely determines its lower adjoint.

Lemma 5.3: Let P and Q be posets with zero, and let $f : P \rightarrow Q$ be a surjective upper adjoint. Let g be the lower adjoint of f. Let 0_P be the zero of P, and 0_Q be the zero of Q. Then $f(0_P) = 0_Q$.

Proof $0_P \leq g(0_Q)$ and hence $f(0_P) \leq fg(0_Q) = 0_Q$. But clearly $0_Q \leq f(0_P)$. ♣

Proposition 5.4: Let P, Q, and R be posets. Let (e, f) be an adjunction between P and Q, and let (g, h) be an adjunction between Q and R. Then (ge, fh) is an adjunction between P and R.

Proof Clearly ge and fh are order-preserving. Let $x \in P$ and $z \in R$. Then $ge(x) \geq z$ if and only if $e(x) \geq h(z)$ if and only if $x \geq fh(z)$. ♣

§ 6. Posets of Mappings

Let P and Q be up-complete posets with zero. Then Q^P denotes the set of all mappings from P to Q. We partially order Q^P as follows. Let $f, g \in Q^P$. Then $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in P$. If $A \subseteq Q^P$ and $x \in P$, then $A(x)$ will denote the set $\{f(x) \mid f \in A\}$.

Lemma 6.1: Let P and Q be posets, and let D be a directed subset of Q^P. Then $D(x)$ is directed for all $x \in P$.

Proof Let $u, v \in D(x)$. Then $u = f(x)$ and $v = g(x)$, where $f, g \in D$. Let $h \in D$ with

$f \leq h$ and $g \leq h$. Then $f(x) \leq h(x)$, $g(x) \leq h(x)$, and $h(x) \in D(x)$. ♣

Proposition 6.2: Let P and Q be up-complete posets with zero. Then Q^P is an up-complete poset with zero.

Proof: Let 0 be the zero of Q. Let $z: P \to Q$ by $z(x) = 0$ for all $x \in P$. Then z is a zero for Q^P. Let D be a directed set in Q^P. Then $D(x)$ is directed for all $x \in P$ by Lemma 2.6.1. Thus we may define $d: P \to Q$ by $d(x) = \sup D(x)$ for all $x \in P$. We will show that $d = \sup D$. Let $f \in D$. Then for $x \in P$, $f(x) \leq \sup D(x) = d(x)$. Thus $f \leq d$, and so d is an upper bound of D. Let g be an upper bound of D. Let $x \in P$. For all $f \in D$, $f \leq g$, and so $f(x) \leq g(x)$. Thus $g(x)$ is an upper bound on $D(x)$, and hence $d(x) = \sup D(x) \leq g(x)$. This inequality holds for all $x \in P$, and so $d \leq g$. Thus $d = \sup D$. ♣

Let P and Q be up-complete posets with zero. Then $[\, P \to Q \,]$ will denote the set of all Scott-continuous mappings from P into Q.

Proposition 6.3: Let P and Q be up-complete posets with zero. Then $[\, P \to Q \,]$ is an up-complete poset with zero.

Proof: Let $z: P \to Q$ be defined as above. Then z is a zero for $[\, P \to Q \,]$. Let D be a directed subset of $[\, P \to Q \,]$. As above, $D(x)$ is directed, and hence we may define $d: P \to Q$ by $d(x) = \sup D(x)$. We will show that $d = \sup D$, where we take the sup in $[\, P \to Q \,]$. Since $d = \sup D$ in Q^P, it suffices to show that $d \in [\, P \to Q \,]$. Thus we must show that d is Scott continuous. Let $x, y \in P$, with $x \leq y$. Let $f \in D$. Then $f(x) \leq f(y) \leq \sup D(y)$. Thus $\sup D(y)$ is an upper bound of $D(x)$, and so $\sup D(x) \leq \sup D(y)$. Thus $d(x) \leq d(y)$, and hence d is order-preserving. Let A be a directed subset of P. Then $d(\sup A)$ is an upper bound of $d(A)$. Let y be an upper bound of $d(A)$. Let $f \in D$. Then for $x \in A$, $f(x) \leq d(x) \leq y$. Thus y is an upper bound of $f(A)$, and so $f(\sup A) = \sup f(A) \leq y$. Hence y is an upper bound of $D(\sup A)$, and so $d(\sup A) \leq y$. Thus $d(\sup A) = \sup d(A)$. ♣

Proposition 6.4: Let P and Q be posets. Let $e: [\, P \to Q \,] \times Q \to Q$ be defined by $e(f, x) = f(x)$. Then e is Scott continuous.

Proof: First we will show that e is order-preserving. Let $(f, x), (g, y) \in [\, P \to Q \,] \times Q$, with $(f, x) \leq (g, y)$. Then $f \leq g$ and $x \leq y$. Thus

$$e(f, x) = f(x) \leq f(y) \leq g(y) = e(g, y).$$

Hence e is order-preserving.

Let D be a directed set in $[\,P \to Q\,] \times Q$. Then $e(\sup D)$ is an upper bound of $e(D)$ by Lemma 2.3.3. Let π_1 and π_2 be the usual projections on $[\,P \to Q\,] \times Q$. Let y be an upper bound for $e(D)$. Let $f \in \pi_1(D)$, and $x \in \pi_2(D)$. Then there exists z such that $(f, z) \in D$ and g such that $(g, x) \in D$. There exists $(h, u) \in D$ such that $(f, z) \leq (h, u)$ and $(g, x) \leq (h, u)$. Now $f(x) \leq f(u) \leq h(u) \leq y$. This holds for all $x \in \pi_2(D)$, and hence y is an upper bound of $f(\pi_2(D))$. Thus $\sup f(\pi_2(D)) \leq y$. But f is Scott continuous, so $f(\sup \pi_2(D)) \leq y$. This holds for all $f \in \pi_1(D)$, so y is an upper bound on $\pi_1(D)(\sup \pi_2(D))$. Thus

$$\left(\sup \pi_1(D)\right)\left(\sup \pi_2(D)\right) = \sup \pi_1(D)\left(\sup \pi_2(D)\right) \leq y,$$

and so

$$e\left(\sup \pi_1(D), \sup \pi_2(D)\right) \leq y.$$

But

$$\sup D = \left(\sup \pi_1(D), \sup \pi_2(D)\right),$$

and so $e(\sup D) \leq y$. Thus $e(\sup D) = \sup e(D)$. ♣

Proposition 6.5: Let P, Q, and R be up-complete posets with zero, and let

$$c\colon [\,P \times Q \to R\,] \to \left(R^Q\right)^P$$

be defined by $c(f)(x)(y) = f(x, y)$ for $f \in [\,P \times Q \to R\,]$, $x \in P$, and $y \in Q$. Then the image of c is contained in $[\,P \to [\,Q \to R\,]\,]$.

Proof: Let $f \in [\,P \times Q \to R\,]$ and $x \in P$. Let D be a directed set in Q. Now $\{x\} \times D$ is a directed set in $P \times Q$, and $\sup(\{x\} \times D) = (x, \sup D)$. Therefore

$$\begin{aligned} c(f)(x)(\sup D) &= f(x, \sup D) \\ &= f\left(\sup(\{x\} \times D)\right) \\ &= \sup f(\{x\} \times D) \\ &= \sup c(f)(x)(D). \end{aligned}$$

Thus $c(f)(x)$ is Scott continuous.

Now we will show that $c(f) \in [\,P \to [\,Q \to R\,]\,]$. Let D' be a directed set in P. Then for all $y \in Q$,

$$c(f)(\sup D')(y) = f(\sup D', y).$$

As before, $D' \times \{y\}$ is a directed set in $P \times Q$, and $\sup D' \times \{y\} = (\sup D', y)$. Thus

$$\begin{aligned} f(\sup D', y) &= f(\sup D' \times \{y\}) \\ &= \sup f(D' \times \{y\}) \\ &= \sup(c(f)(D')(y)) \\ &= \sup(c(f)(D'))(y). \end{aligned}$$

Thus for all $y \in Q$,

$$c(f)(\sup D')(y) = (\sup c(f)(D'))(y),$$

and so

$$c(f)(\sup D') = \sup c(f)(D').$$

Hence $c(f)$ is Scott continuous. ♣

Proposition 6.6: Let P, Q, and R be up-complete posets with zero, and let

$$c' : [\, P \to [\, Q \to R \,]\,] \to R^{P \times Q}$$

be defined by $c'(f)(x, y) = f(x)(y)$ for $f \in [\, P \to [\, Q \to R \,]\,]$, $x \in P$, and $y \in Q$. Then the image of c' is contained in $[\, P \times Q \to R \,]$.

Proof: Let $f \in [\, P \to [\, Q \to R \,]\,]$. Let D be a directed subset of $P \times Q$. Then

$$\sup D = (\sup \pi_1(D), \sup \pi_2(D)).$$

Thus

$$c'(f)(\sup D) = f(\sup \pi_1(D))(\sup \pi_2(D)).$$

Now

$$c'(f)(D) = \{\, c'(f)(x, y) \mid (x, y) \in D \,\} = \{\, f(x)(y) \mid (x, y) \in D \,\}.$$

Thus we must show that

$$f(\sup \pi_1(D))(\sup \pi_2(D)) = \sup\{\, f(x)(y) \mid (x, y) \in D \,\}.$$

Let $(x, y) \in D$. Then $x \leq \sup \pi_1(D)$, and so $f(x) \leq f(\sup \pi_1(D))$. Now $y \leq \sup \pi_2(D)$, and so

$$f(x)(y) \leq f(x)(\sup \pi_2(D)) \leq f(\sup \pi_1(D))(\sup \pi_2(D)).$$

Thus $f(\sup \pi_1(D))(\sup \pi_2(D))$ is an upper bound on $\{\, f(x)(y) \mid (x, y) \in D \,\}$.

Let t be an upper bound on $\{ f(x)(y) \mid (x, y) \in D \}$. Let $x \in \pi_1(D)$. Then $(x, v) \in D$ for some $v \in Q$. If $y \in \pi_2(D)$, then $(u, y) \in D$ for some $u \in P$. There exist $w \in P$ and $z \in Q$ such that $(x, v) \leq (w, z)$ and $(u, y) \leq (w, z)$. Then $x \leq w$ and $y \leq z$. Thus $f(x) \leq f(w)$, and so

$$f(x)(y) \leq f(w)(y) \leq f(w)(z) \leq t.$$

Hence t is an upper bound on $f(x)\big(\pi_2(D)\big)$. Thus

$$f(x)\big(\sup \pi_2(D)\big) = \sup f(x)\big(\pi_2(D)\big) \leq t.$$

Now f is Scott continuous, and so

$$f\big(\sup \pi_1(D)\big) = \sup f\big(\pi_1(D)\big).$$

Thus

$$\begin{aligned} f\big(\sup \pi_1(D)\big)\big(\sup \pi_2(D)\big) &= \Big(\sup f\big(\pi_1(D)\big)\Big)\big(\sup \pi_2(D)\big) \\ &= \sup\Big(f\big(\pi_1(D)\big)\big(\sup \pi_2(D)\big)\Big). \end{aligned}$$

But t is an upper bound on $f\big(\pi_1(D)\big)\big(\sup \pi_2(D)\big)$, and so

$$\sup\Big(f\big(\pi_1(D)\big)\big(\sup \pi_2(D)\big)\Big) \leq t.$$

Thus

$$f\big(\sup \pi_1(D)\big)\big(\sup \pi_2(D)\big) \leq t.$$

We have shown that

$$f\big(\sup \pi_1(D)\big)\big(\sup \pi_2(D)\big) = \sup\{ f(x)(y) \mid (x, y) \in D \}.$$

Therefore $c'(f)$ is Scott continuous. ♣

Proposition 6.7: Let P, Q, and R be up-complete posets with zero. Then the maps

$$c: [\, P \times Q \to R \,] \to [\, P \to [\, Q \to R \,]\,]$$

and

$$c': [\, P \to [\, Q \to R \,]\,] \to [\, P \times Q \to R \,]$$

defined above are order isomorphisms, and $c' = c^{-1}$.

Proof: First we will show that c is order-preserving. Let $f, g \in [\, P \times Q \to R \,]$, with $f \leq g$. Let $x \in P$ and $y \in Q$. Then

$$(f)(x)(y) = f(x, y) \leq g(x, y) = c(g)(x)(y).$$

The inequality holds for all $y \in Q$, and so $c(f)(x) \leq c(g)(x)$. This inequality holds for all $x \in P$, and therefore $c(f) \leq c(g)$.

Next we will show that c' is order preserving. Let $f, g \in [\ P \rightarrow [\ Q \rightarrow R\ ]\ ]$, with $f \leq g$. Then $f(x) \leq g(x)$. Thus

$$c'(f)(x, y) = f(x)(y) \leq g(x)(y) = c'(g)(x, y).$$

This inequality holds for all $(x, y) \in P \times Q$, and therefore $c'(f) \leq c'(g)$.

Finally, we will show that $c' = c^{-1}$. Let $f \in [\ P \times Q \rightarrow R\ ]$. Then

$$c'c(f)(x, y) = c(f)(x)(y) = f(x, y)$$

for all $(x, y) \in P \times Q$. Thus $c'c(f) = f$. Let $g \in [\ P \rightarrow [\ Q \rightarrow R\ ]\ ]$. Let $x \in P$ and $y \in Q$. Then

$$cc'(g)(x)(y) = c'(g)(x, y) = g(x)(y).$$

The equality holds for all $y \in Q$, and so $cc'(g)(x) = g(x)$. But this equality holds for all $x \in P$, and therefore $cc'(g) = g$. ♣

Proposition 6.8: Let P, Q, and R be up-complete posets with zero. Define C: $[\ P \rightarrow Q\ ] \times [\ Q \rightarrow R\ ] \rightarrow [\ P \rightarrow R\ ]$ by $C(f, g) = gf$ for $f \in [\ P \rightarrow Q\ ]$ and $g \in [\ Q \rightarrow R\ ]$. Then C is Scott continuous.

Proof: First we will show that C is order-preserving. Let $(e, f), (g, h) \in [\ P \rightarrow Q\ ] \times [\ Q \rightarrow R\ ]$, with $(e, f) \leq (g, h)$. Then

$$C(e, f)(x) = ef(x) \leq hf(x) \leq hg(x) = C(g, h)(x).$$

The inequalities hold for all $x \in D$, and so $C(e, f) \leq C(g, h)$.

Let D be a directed set in $[\ P \rightarrow Q\ ] \times [\ Q \rightarrow R\ ]$. Since C is order-preserving, $C(\sup D)$ is an upper bound of $C(D)$. Now $\sup D = \left(\sup \pi_1(D), \sup \pi_2(D)\right)$. Thus

$$C(\sup D) = \left(\sup \pi_2(D)\right) \circ \left(\sup \pi_1(D)\right)$$

. Let $h \in [\ P \rightarrow R\ ]$ be an upper bound of $C(D)$. Let $f \in \pi_1(D)$ and $g \in \pi_2(D)$. Then there exist $a \in \pi_1(D)$ and $b \in \pi_2(D)$ such that $(f, b) \in D$ and $(a, g) \in D$. There exists $(c, d) \in D$ such that $(f, b) \leq (c, d)$ and $(a, g) \leq (c, d)$. Thus $g \leq d$ and $f \leq c$. For all $x \in D$, $gf(x) \leq df(x) \leq dc(x)$. Hence $gf \leq dc \leq h$. Thus h is an upper bound on $C\left(\pi_1(D) \times \{g\}\right)$.

Let $x \in P$. Then

$$\begin{aligned} g \circ (\sup \pi_1(D))(x) &= g\big((\sup \pi_1(D))(x)\big) \\ &= g\big(\sup(\pi_1(D)(x))\big) \\ &= \sup g\pi_1(D)(x) \\ &= \sup C(\pi_1(D) \times \{g\})(x) \\ &= \big(\sup C(\pi_1(D) \times \{g\})\big)(x). \end{aligned}$$

The equalities hold for all $x \in P$, and so $g \circ \sup \pi_1(D) = \sup C(\pi_1(D) \times \{g\}) \leq h$. This inequality holds for all $g \in [\,Q \to R\,]$, and so h is an upper bound on $C(\pi_2(D) \times \{\sup \pi_2(D)\})$.

Let $x \in P$. Then

$$\begin{aligned} (\sup \pi_2(D)) \circ (\sup \pi_1(D))(x) &= (\sup \pi_2(D)) \circ (\sup \pi_1(D))(x) \\ &= \sup\big(\pi_2(D)(\sup \pi_1(D)(x))\big) \\ &= \sup\big(C(\pi_2(D) \times \{\sup \pi_1(D)\})(x)\big) \\ &= \big(\sup C(\pi_2(D) \times \{\sup \pi_1(D)\})\big)(x). \end{aligned}$$

These equalities hold for all $x \in P$, and so

$$(\sup \pi_1(D)) \circ (\sup \pi_2(D)) = \sup C(\pi_2(D) \times \{\pi_1(D)\}) \leq h.$$

Therefore

$$C(\sup D) = (\sup \pi_2(D)) \circ (\sup \pi_1(D)) = \sup C(D). \clubsuit$$

§ 7. The Category U

Let **U** denote the category whose objects are up-complete posets with zero, and whose morphisms are surjective Scott-continuous upper adjoints. For a morphism $f: P \to Q$, we denote the lower adjoint of f by $\hat{f}$. Note that by Proposition 2.5.1, lower adjoints between up-complete posets with zero are always Scott-continuous.

Proposition 7.1: Let $\{P_i\}_i \in I$ be a family of up-complete posets with zero. Then the usual projections $\pi_j: \prod_{i \in I} P_i \to P_j$ are surjective Scott-continuous upper adjoints.

Proof: We have shown that the π_j are Scott-continuous. Let 0_j be the zero of P_j for each $j \in I$. Define $e_j: P_j \to \prod_{i \in I} P_i$ by

$$\pi_k e_j(s) = \begin{cases} 0_k, & \text{if } k \neq j; \\ s, & \text{if } k = j; \end{cases}$$

for $j, k \in I$ and $s \in P_j$. Then e_j is order-preserving. Let I_j be the identity map on P_j, and let I be the identity map on $\prod_{i \in I} P_i$. Then $\pi_j e_j(s) = s$ for all $s \in P_j$, and so $\pi_j e_j = I_j$. Let $x \in \prod_{i \in I} P_i$. Then

$$\pi_k e_j \pi_j(x) = \begin{cases} 0_k, & \text{if } k \neq j; \\ \pi_k(x), & \text{if } k = j. \end{cases}$$

Thus $\pi_k e_j \pi_j(x) \leq \pi_k(x)$ for all $k \in I$, and therefore $e_j \pi_j(x) \leq x$. Hence $e_j \pi_j \leq I$. Thus π_j is a surjective upper adjoint, and $e_j = (\pi_j)^{\wedge}$. ♣

A projective system in U is a family $\{P_i\}_{i \in I}$ over a directed poset I of up-complete posets with zero equipped with a collection $\{p_i^j\}_{i,j \in I, i \leq j}$ of morphisms such that

(1) $p_i^j : P_j \to P_i$ for all $i, j \in I$ with $i \leq j$.

(2) p_i^i is the identity map on P_i for all $i \in I$.

(3) the following diagram commutes for all $i, j, k \in I$ with $i \leq j \leq k$.

$$\begin{tikzcd} P_k \arrow[r, "p_j^k"] \arrow[d, equal] & P_j \arrow[d, "p_i^j"] \\ P_k \arrow[r, "p_j^k"] & P_i \end{tikzcd}$$

The p_i^j are called bonding maps.

Proposition 7.2: Let $\{P_i\}_{i \in I}$ be a projective system in U with bonding maps $\{p_i^j\}_{i,j \in I, i \leq j}$. We define $\varprojlim P_i$ by

$$\varprojlim P_i = \left\{ x \in \prod_{i \in I} P_i \;\middle|\; p_i^j \pi_j(x) = \pi_i(x) \text{ for all } i, j \in I, i \leq j \right\}.$$

Let $\pi_j : \varprojlim P_i \to P_j$ be the restriction of the usual projection map on $\prod_{i \in I} P_i$ for each $j \in I$. Then $\varprojlim P_i$ is an up-complete poset with zero, and each π_j is a surjective Scott continuous upper adjoint.

Proof: Let 0_j be the zero of P_j for each $j \in I$. If we define $0 \in \prod_{i \in I} P_i$ by $\pi_j(0) = 0_j$ for all $j \in I$, then $0 \in \varprojlim P_i$ by Lemma 2.5.3. Clearly 0 is a zero for $\varprojlim P_i$. Let D be a directed set in $\varprojlim P_i$. Define $d \in \prod_{i \in I} P_i$ by $\pi_j(d) = \sup \pi_j(D)$ for all $j \in I$. We show that d is the supremum of D in $\varprojlim P_i$. Since d is the supremum of D in $\prod_{i \in I} P_i$, it suffices to show that

$d \in \varprojlim P_i$. Now

$$p_i^j(d) = p_i^j\left(\sup \pi_j(D)\right) = \sup p_i^j \pi_j(D) = \sup \pi_j(D) = \pi_i(D).$$

Thus $d = \sup D$, and so $\varprojlim P_i$ is an up-complete poset with zero.

We have also shown that $\sup \pi_j(D) = \pi_j(\sup D)$ for every directed set $D \subseteq \varprojlim P_i$. Therefore π_j is Scott continuous. We now show that π_j is a surjective upper adjoint. We define $e_j : \mathbf{P}_j \to \varprojlim \mathbf{P}_i$ as follows:

$$\pi_k e_j(s) = p_k^l (p_j^l)^\wedge (s) \text{ where } j \leq l \text{ and } k \leq l$$

for all $j, k \in I$ and $s \in P_j$. We must show that e_j is well defined. Let $l, m \in I$, $j \leq m$, $j \leq l$, $j \leq m$, and $k \leq m$. Then there exists $n \in I$ such that $l \leq n$ and $m \leq n$. Now

$$\begin{aligned} p_k^m (p_j^m)^\wedge (s) &= p_k^m p_m^n (p_m^n)^\wedge (p_j^m)^\wedge (s) \\ &= p_k^n (p_j^n)^\wedge (s) \\ &= p_k^l p_l^n (p_l^n)^\wedge (p_j^l)^\wedge (s) \\ &= p_k^l (p_j^l)^\wedge (s). \end{aligned}$$

Thus e_j is well defined. Note that for $k \leq j$ we may take $l = j$, and so

$$\pi_k e_j(s) = p_l^j (p_j^j)^\wedge (s) = p_k^j(s).$$

If $j \leq k$, then we may take $l = k$, and so

$$\pi_k e_j(s) = p_k^k (p_j^k)^\wedge (s) = (p_j^k)^\wedge (s).$$

Now we show that e_j is order-preserving. Let $s, t \in P_j$, with $s \leq t$. Let $k \in I$. Choose $l \in I$ such that $j, k \leq l$. Then

$$\pi_k e_j(s) = p_k^l (p_j^l)^\wedge (s) \leq p_k^l (p_j^l)^\wedge (t) = \pi_k e_j(t).$$

The inequality holds for all $k \in I$, and so $e_j(s) \leq e_j(t)$. Therefore e_j is order-preserving.

Now we show that e_j is a lower adjoint for π_j. Let $s \in P_j$. Then $\pi_j e_j(s) = p_j^j (p_j^j)^\wedge (s) = s$. Thus $\pi_j e_j = I_j$. Let $x \in \varprojlim P_i$, and let $k \in I$. Choose l such that $j \leq l$ and $k \leq l$. Then

$$\begin{aligned} \pi_k e_j \pi_j(x) = p_k^l (p_j^l)^\wedge \pi_j(x) &= p_k^l (p_j^l)^\wedge p_j^l \pi_l(x) \\ &= p_k^l \pi_l(x) = \pi_k(x). \end{aligned}$$

Thus $e_j \pi_j(x) \leq x$, and therefore $e_j \pi_j \leq I$. Thus π_j is a surjective upper adjoint, and $e_j = (\pi_j)^\wedge$. ♣

Lemma 7.8: Let $\{P_i\}_{i\in I}$ be a projective system in U with bonding maps

$$\{ p_i^j \}_{i,j\in I, i\leq j}.$$

Let $x \in \varprojlim P_i$. Then $x = \sup_{i\in I} (\pi_i)^{\wedge} \pi_i(x)$.

Proof: $(\pi_i)^{\wedge} \pi_i(x) \leq x$, so x is an upper bound of $\{ (\pi_i)^{\wedge} \pi_i(x) \}_{i\in i}$. Let y be an upper bound. Then for $j \in I$, $(\pi_j)^{\wedge} \pi_j(x) \leq y$, so $\pi_j(\pi_j)^{\wedge} \pi_j(x) \leq \pi_j(y)$. Thus $\pi_j(x) \leq \pi_j(y)$. This inequality holds for all $j \in I$, and so $x \leq y$. Thus $x = \sup_{i\in I} (\pi_i)^{\wedge} \pi_i(x)$. ♣

§ 8. Fixed Points of Functors

Let $\mathcal{F}$ be a self-functor of $\mathcal{U}$. In this section we show how one may construct "fixed points" for $\mathcal{F}$ (assuming that $\mathcal{F}$ satisfies certain technical conditions). By a fixed point for $\mathcal{F}$ we mean an up-complete poset with zero P such that P is isomorphic to $\mathcal{F}(P)$.

Let P be an up-complete poset with zero, and assume that there exists a surjective Scott continuous upper adjoint $f: \mathcal{F}(P) \to P$. Then we construct a projective system indexed by the non-negative integers. We define a sequence $\{P_i\}_{i\geq 0}$ as follows:

$$\begin{cases} P_0 = P \\ P_{j+1} = \mathcal{F}(P_j) & \text{if } j > 0. \end{cases}$$

We define a sequence $\{f_i: P_{i+1} \to P_i\}_{i\geq 0}$ as follows:

$$\begin{cases} f_0 = f \\ f_{j+1} = \mathcal{F}(f_j) & \text{if } j \geq 0. \end{cases}$$

Let I_j be the identity map on P_j for each $j \geq 0$. We define a system $\{ f_i^j : P_j \to P_i \}_{0\leq i\leq j}$ as follows:

$$\begin{cases} f_i^i = I_i & \text{if } i \geq 0 \\ f_i^j = f_i f_{i+1} \cdots f_{j-1} & \text{if } 0 \leq i < j. \end{cases}$$

Then $\{P_i\}_{i\geq 0}$ is a projective system in $\mathcal{U}$ with bonding maps $\{ f_i^j \}_{0\leq i\leq j}$. Note that $\mathcal{F}(f_i^j) = f_{i+1}^{j+1}$.

Before proceeding to the main theorem we prove two extremely useful lemmas.

Lemma 8.1: Let P be a up-complete poset with zero. Let I be a directed set, and let $\{ x_i^j \}_{i,j\in I}$ be subset of P such that

(1) for all $i, j, k \in I$, $j \leq k$ implies $x_{i,j} \leq x_{i,k}$, and

(2) for all $i, j, k \in I$, $i \leq k$ implies $x_{i,j} \leq x_{k,j}$.

Then

$$\sup_{i,j\in I} x_{i,j} = \sup_{i\in I} x_{i,i}$$
$$= \sup_{i\in I}\big(\sup_{j\in I} x_{i,j}\big)$$
$$= \sup_{j\in I}\big(\sup_{i\in I} x_{i,j}\big).$$

Proof: Let $k \in I$. Then $x_{k,k} \leq \sup_{i,j\in I} x_{i,j}$. This holds for all $k \in I$, and so $\sup_{i\in I} x_{i,i} \leq \sup_{i,j\in I} x_{i,j}$. Let $k, l \in I$. Then there exists $m \in I$ such that $k \leq m$ and $l \leq m$. Then

$$x_{k,l} \leq x_{m,l} \leq x_{m,m} \leq \sup_{i\in I} x_{i,i}.$$

Thus $\sup_{i,j\in I} x_{i,j} \leq \sup_{i\in I} x_{i,i}$. Therefore $\sup_{i,j\in I} x_{i,j} = \sup_{i\in I} x_{i,i}$.

Now

$$x_{k,l} \leq \sup_{j\in I} x_{k,j} \leq \sup_{i\in I}\big(\sup_{j\in I} x_{i,j}\big)$$

for all $k, l \in I$, and so

$$\sup_{i,j\in I} x_{i,j} \leq \sup_{i\in I}\big(\sup_{j\in I} x_{i,j}\big).$$

Also $x_{k,l} \leq \sup_{i,j\in I} x_{i,j}$ for all $l \in I$, and so

$$\sup_{j\in I} x_{k,j} \leq \sup_{i,j\in I} x_{i,j}.$$

But this holds for all $k \in I$, and so

$$\sup_{i\in I}\big(\sup_{j\in I} x_{i,j}\big) \leq \sup_{i,j\in I} x_{i,j}.$$

Thus

$$\sup_{i\in I}\big(\sup_{j\in I} x_{i,j}\big) = \sup_{i,j\in I} x_{i,j}.$$

Similarly

$$\sup_{j\in I}\big(\sup_{i\in I} x_{i,j}\big) = \sup_{i,j\in I} x_{i,j}. \clubsuit$$

Lemma 3.2: Let I be a directed set. Let $P_1, \ldots, P_{n+1}$ be up-complete posets with zero. For $0 \leq i \leq n$ let $\{f_{i,j}: P_{i+1} \to P_i\}_{j\in I}$ be a collection of Scott continuous maps such that $f_i^j \leq f_i^k$ whenever $j, k \in I$, $j \leq k$, and $0 \leq i \leq n$, and let $f_i = \sup_{j\in I} f_i^j$. Then $f_1 \circ \ldots \circ f_n = \sup_{j\in I} f_1^j \circ \ldots \circ f_n^j$.

Proof: First we treat the case $n=2$. Let $x \in P$. Then

$$
\begin{aligned}
f_1 \circ f_2(x) &= f_1 \circ \big(\sup_{j\in I} f_2^j\big)(x) \\
&= f_1\big(\sup_{j\in I} f_2^j(x)\big) \\
&= \sup_{j\in I} f_1 f_2^j(x) \\
&= \sup_{j\in I}\big(\sup_{k\in I} f_1^k\big) \circ f_2^j(x) \\
&= \sup_{j\in I}\big(\sup_{k\in I} f_1^k f_2^j(x)\big) \\
&= \sup_{j\in I} f_1^j f_2^j(x) \qquad (\text{ by Lemma 2.8.1 }) \\
&= \big(\sup_{j\in I} f_1^j f_2^j\big)(x).
\end{aligned}
$$

Thus $f_1 \circ f_2 = \sup_{j\in I} f_1^j f_2^j$.

Now we precede by induction. Assume that $f_1 \circ \ldots f_{n-1} = \sup_{j\in I} f_1^j \circ \ldots \circ f_{n-1}^j$. For each $j \in I$, define $F_j : P_n \to P_1$ by $F_j = f_1^j \circ \ldots \circ f_{n-1}^j$. Then $F_j \leq F_k$ for $j, k \in I$, and $j \leq k$. Define $F : P_n \to P_1$ by $F = f_1 \circ \ldots \circ f_{n-1}$. Then

$$
\begin{aligned}
F &= \sup_{j\in I} f_1^j \circ \ldots \circ f_{n-1}^j \qquad (\text{ by the induction hypothesis }) \\
&= \sup_{j\in I} F_j.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f_1 \circ \ldots \circ f_n &= F \circ f_n \\
&= \sup_{j\in I} F_j \circ f_n^j \qquad (\text{ since the lemma holds in the case } n=2 \text{ }) \\
&= \sup_{j\in I} f_i^j \circ \ldots \circ f_n^j. \clubsuit
\end{aligned}
$$

Theorem 8.8: Let P be an up-complete poset with zero. Let $\mathcal{F}$ be a self functor of $\mathcal{U}$ such that $\mathrm{Mor}([\,P \to P\,], P) \neq \emptyset$. Define $\{P_i\}_{i\geq 0}$ and $\{\,f_i^j\,\}_{0\leq i\leq j}$ as above, where f_0 is chosen in $\mathrm{Mor}([\,P \to P\,], P)$. Let $\pi_j : \varprojlim P_i \to P_j$ be the usual projection for each $j \geq 0$. Let I be the identity map on $\varprojlim P_i$, and let I' be the identity map on $\mathcal{F}(\varprojlim P_i)$. Assume that $\sup_{i\geq 0}(\mathcal{F}(\pi_i))^{\wedge}\, \mathcal{F}(\pi_i) = I'$. Then $\mathcal{F}(\varprojlim P_i)$ is order isomorphic to $\varprojlim P_i$.

Proof: Consider the diagram below:

$$\begin{array}{ccc}
\varprojlim P_i & & \mathcal{F}\left(\varprojlim P_i\right) \\
(\pi_{j+1})^{\wedge} \Big\uparrow \Big\downarrow \pi_{j+1} & & (\mathcal{F}(\pi_j))^{\wedge} \Big\uparrow \Big\downarrow \mathcal{F}(\pi_j) \\
P_{j+1} & = & \mathcal{F}(P_j)
\end{array}$$

For $j \geq 0$, we define $\Phi_j : \varprojlim P_i \to \mathcal{F}\left(\varprojlim P_i\right)$ by $\Phi_j = (\mathcal{F}(\pi_j))^{\wedge}\, \pi_{j+1}$. Similarly we define $\Psi_j : \mathcal{F}\left(\varprojlim P_i\right) \to \varprojlim P_i$ by $\Psi_j = (\pi_{j+1})^{\wedge}\, \mathcal{F}(\pi_j)$. Let $0 \leq i \leq j$. Then

$$\begin{aligned}
\Phi_i &= (\mathcal{F}(\pi_i))^{\wedge}\, \pi_{i+1} \\
&= \left(\mathcal{F}(f_i^j \pi_j)\right)^{\wedge} f_{i+1}^{j+1} \pi_{j+1} \\
&= \left(\mathcal{F}(f_i^j)\, \mathcal{F}(\pi_j)\right)^{\wedge} f_{i+1}^{j+1} \pi_{j+1} \\
&= (\mathcal{F}(\pi_j))^{\wedge} \left(\mathcal{F}(f_i^j)\right)^{\wedge} f_{i+1}^{j+1} \pi_{j+1} \\
&= (\mathcal{F}(\pi_j))^{\wedge} (f_{i+1}^{j+1})^{\wedge} f_{i+1}^{j+1} \pi_{j+1} \\
&\leq (\mathcal{F}(\pi_j))^{\wedge}\, \pi_{j+1} \\
&= \Phi_j,
\end{aligned}$$

and

$$\begin{aligned}
\Psi_i &= (\pi_{i+1})^{\wedge}\, \mathcal{F}(\pi_i) \\
&= (f_{i+1}^{j+1} \pi_{j+1})^{\wedge}\, \mathcal{F}(f_i^j \pi_j) \\
&= (\pi_{j+1})^{\wedge} (f_{i+1}^{j+1})^{\wedge}\, \mathcal{F}(f_i^j)\, \mathcal{F}(\pi_j) \\
&= (\pi_{j+1})^{\wedge} (f_{i+1}^{j+1})^{\wedge} f_{i+1}^{j+1}\, \mathcal{F}(\pi_j) \\
&\leq (\pi_{j+1})^{\wedge}\, \mathcal{F}(\pi_j) \\
&= \Psi_j.
\end{aligned}$$

Thus $\{\Phi_i\}_{i \geq 0}$ and $\{\Psi_j\}_{j \geq 0}$ are increasing sequences. We may therefore define

$$\Phi \in [\, \varprojlim P_i \to \mathcal{F}\left(\varprojlim P_i\right) \,]$$

by $\Phi = \sup_{i \geq 0} \Phi_i$, and

$$\Psi \in [\, \mathcal{F}\left(\varprojlim P_i\right) \to \varprojlim P_i \,]$$

by $\Psi = \sup_{i \geq 0} \Psi_i$. We show that $\Psi = \Phi^{-1}$. Now

$$\begin{aligned}
\Psi\Phi &= \sup_{i \geq 0} \Psi_i \Phi_i \\
&= \sup_{i \geq 0} (\pi_{i+1})^{\wedge} \mathcal{F}(\pi_i)(\mathcal{F}(\pi_i))^{\wedge} \pi_{i+1} \\
&= \sup_{i \geq 0} (\pi_{i+1})^{\wedge} \pi_{i+1} \\
&= \mathcal{I} \qquad (\text{ by Lemma 2.7.3 }),
\end{aligned}$$

and

$$\begin{aligned}
\Phi\Psi &= \sup_{i \geq 0} \Phi_i \Psi_i \\
&= \sup_{i \geq 0} (\mathcal{F}(\pi_i))^{\wedge} \pi_{i+1}(\pi_{i+1})^{\wedge} \mathcal{F}(\pi_i) \\
&= \sup_{i \geq 0} (\mathcal{F}(\pi_i))^{\wedge} \mathcal{F}(\pi_i) \\
&= \mathcal{I}'.
\end{aligned}$$

Thus $\Psi = \Phi^{-1}$. ♣

Chapter 3

Models for the Lambda Calculus

We now apply the results of the preceding chapter to obtain models for the lambda calculus. The basic idea is as follows. The correspondence $P \rightarrow [\ P \rightarrow P\ ]$ defines a self-functor of $\mathcal{U}$ which satisfies the hypotheses of Theorem 2.8.3. We therefore obtain a fixed point P_∞ of this functor. Now P_∞ is isomorphic to $[\ P_\infty \rightarrow P_\infty\ ]$, and by Theorem 1.3.1, P_∞ is a model for the lambda calculus. We show that the correspondence $P \rightarrow P_\infty$ defines a monofunctor, and finally, calculate the elements of P_∞ corresponding to several combinators.

§ 1. The Construction of Models

We define a self-functor S of $\mathbf{U}$ as follows:

(1) For an up-complete poset with zero P, let $S(P) = [\, P \to P \,]$.

(2) Let P and Q be up-complete posets with zero. Then for a surjective Scott continuous upper adjoint $f: P \to Q$, define $S(f): S(P) \to S(Q)$ by $S(f)(\phi) = f\phi\hat{f}$ for $\phi \in S(P)$.

Lemma 1.1: Let P and Q be up-complete posets with zero. Let $f: P \to Q$ be a surjective Scott continuous upper adjoint. Then $S(f): S(P) \to S(Q)$ is a surjective Scott continuous upper adjoint, and $(S(f))^{\wedge}(\psi) = \hat{f}\psi f$ for all $\psi \in S(Q)$.

Proof: Define

$$\alpha: [\, P \to P \,] \to [\, P \to Q \,] \times [\, P \to P \,]$$

by $\alpha(\phi) = (f, \phi)$ for $\phi \in [\, P \to P \,]$. Then α is Scott continuous by Lemma 2.4.2. Define

$$\beta: [\, P \to Q \,] \times [\, P \to P \,] \to [\, P \to Q \,]$$

by $\beta(\phi, \psi) = \psi\phi$ for $\psi \in [\, P \to Q \,]$ and $\phi \in [\, P \to P \,]$. Then β is Scott continuous by Proposition 2.6.8. Define

$$\gamma: [\, P \to Q \,] \to [\, P \to Q \,] \times [\, Q \to Q \,]$$

by $\gamma(\psi) = (\psi, \hat{f})$ for $\psi \in [\, P \to Q \,]$. Then γ is Scott continuous by Lemma 2.4.2. Define

$$\delta: [\, P \to Q \,] \times [\, Q \to Q \,] \to [\, P \to Q \,]$$

by $\delta(\psi, \rho) = \rho\gamma$ for $\psi \in [\, P \to Q \,]$ and $\rho \in [\, Q \to Q \,]$. Then δ is Scott continuous by Proposition 2.6.8. Let $\phi \in [\, P \to Q \,]$. Then

$$\delta\gamma\beta\alpha(\phi) = \delta\alpha\beta(\phi, f) = \delta\gamma(f\phi) = \delta(f\phi\hat{f}) = f\phi\hat{f} = S(f)(\phi).$$

Therefore $S(f) = \delta\gamma\beta\alpha$, and so $S(f)$ is Scott continuous. Define $G: S(Q) \to S(Q)$ by $G(\psi) = \hat{f}\psi f$ for $\psi \in S(Q)$. Then G is order-preserving. Furthermore,

$$\big(S(f) \circ G\big)(\psi) = S(f)(\hat{f}\psi f) = f\hat{f}\psi f\hat{f} = \psi$$

for all $\psi \in S(Q)$, and

$$\Big(G \circ \big(S(f)\big)\Big)(\phi) = G(f\phi\hat{f}) = \hat{f}f\phi\hat{f}f \leq \phi$$

for all $\phi \in S(P)$. Thus $S(f)$ is a surjective upper adjoint, and $G = \big(S(f)\big)^{\wedge}$. ♣

Lemma 1.2: Let P, Q, and R be up-complete posets with zero. Let I be the identity map on P, and let I' be the identity map on $S(P)$. Then $S(I) = I'$. Let $f: P \to Q$ and $g: Q \to R$ be surjective Scott continuous upper adjoints. Then $S(gf) = S(g) \circ S(f)$.

Proof: Let $\phi \in S(P)$. Then $S(I)(\phi) = I\phi\hat{I} = \phi$. Thus $S(I) = I'$.

Let $\phi \in S(P)$. Then

$$\big(S(gf)\big)(\phi) = gf\phi\widehat{(gf)} = gf\phi\hat{f}\hat{g} = \big(S(g)\big)(f\phi\,\hat{f}) = \big(S(g)\big) \circ \big(S(f)\big)(\phi).$$

Thus $S(gf) = \big(S(g)\big) \circ \big(S(f)\big)$. ♣

The two preceding lemmas show that S is indeed a functor. We show next that if P is an up-complete poset with zero then $\mathrm{Mor}(S(P), P)$ is never empty.

Lemma 1.3: Let P be an up-complete poset with zero. Let 0 be the zero of P. Define $e_0: S(P) \to P$ by $e_0(\phi) = \phi(0)$ for $\phi \in S(P)$. Then e_0 is a surjective Scott continuous upper adjoint.

Proof: Define

$$\alpha: [\, P \to P \,] \to [\, P \to P \,] \times P$$

by $\alpha(\phi) = (\phi, 0)$ for $\phi \in [\, P \to P \,]$. Then α is Scott continuous by Lemma 2.4.2. Define $\beta: [\, P \to P \,] \times P \to P$ by $\beta(\phi, x) = \phi(x)$ for $\phi \in [\, P \to P \,]$ and $x \in P$. Then β is Scott continuous by Proposition 2.6.3. Now

$$\beta\alpha(\phi) = \beta(\phi, 0) = \phi(0) = e_0(\phi)$$

for $\phi \in S(P)$, and so $e_0 = \beta\alpha$. Thus e_0 is Scott continuous. Define $k: P \to S(P)$ by $k(x)(y) = x$ for $x, y \in P$. Then k is order-preserving. For $x \in P$, $e_0 k(x) = k(x)(0) = x$. For $\phi \in S(P)$ and $x \in P$,

$$(ke_0(\phi))(x) = e_0(\phi) = \phi(0) \leq \phi(x).$$

This inequality holds for all $x \in P$, and so $ke_0(\phi) \leq \phi$. Thus e_0 is a surjective upper adjoint, and $k = \widehat{(e_0)}$. ♣

Theorem 1.4: Let P be an up-complete poset with zero. We define a sequence $\{P_i\}_{i \geq 0}$ of up-complete posets with zero by

$$P_i = \begin{cases} P & \text{if } i = 0; \\ [\, P_{i-1} \to P_{i-1} \,] & \text{if } i \geq 0. \end{cases}$$

We define a sequence $\{e_i : P_{i+1} \to P_i\}_{i \geq 0}$ of maps as follows. Let $\phi \in P_{i+1}$. Then

$$e_i(\phi) = \begin{cases} \phi(0) & \text{if } i = 0; \\ e_{i-1}\phi(e_{i-1})^{\wedge} & \text{if } i > 0. \end{cases}$$

Let I_i be the identity map on P_i for $i \geq 0$. We define a system of maps $\{ e_i^j : P_j \to P_i \}_{0 \leq i \leq j}$ by

$$e_i^j = \begin{cases} I_i & \text{if } i = j; \\ \mathrm{P}_i \mathrm{P}_{i+1} \cdots P_{j-1} & \text{if } i < j. \end{cases}$$

We define $P_\infty = \varprojlim P_i$, and $\Phi_i : P_\infty \to [\, P_\infty \to P_\infty \,]$ by

$$\Phi_i(x) = (\, S(\pi_i))^{\wedge}\, \pi_{i+1}(x) = (\pi_i)^{\wedge} \circ (\, \pi_{i+1}(x)) \circ \pi_i$$

$$\Psi(\phi) = (\pi_{i+1})^{\wedge} \circ (\, S(\pi_i))\, (\phi) = (\pi_{i+1})^{\wedge} (\, \pi_i \phi(\pi_i)^{\wedge}\,).$$

We define $\Phi = \sup_{i>0} \Phi_i$ and $\Psi = \sup_{i>0} \Psi_i$. Then Φ is an order isomorphism and $\Psi = \Phi^{-1}$.

We define $\Phi = \sup_{i \geq 0} \Phi_i$ and $\Psi = \sup_{i \geq 0} \Psi_i$. Then Φ is an order isomorphism and $\Psi = \Phi^{-1}$.

Proof: By Theorem 2.8.3, it suffices to show that $\sup_{i \geq 0} (\, S(\pi_i))^{\wedge}\, S(\pi_i)$ is the identity map on $[\, P_\infty \to P_\infty \,]$. Let $\phi \in [\, P_\infty \to P_\infty \,]$. Then

$$\begin{aligned} \sup_{i \geq 0} (\, S(\pi_i))^{\wedge} \circ (\, S(\pi_i))\, (\phi) &= \sup_{i \geq 0} (\, S(\pi_i))^{\wedge} (\, \pi_i \phi(\pi_i)^{\wedge}\,) \\ &= \sup_{i \geq 0} (\pi_i)^{\wedge}\, \pi_i \phi(\pi_i)^{\wedge}\, \pi_i \\ &= (\, \sup_{i \geq 0} (\pi_i)^{\wedge}\, \pi_i) \circ \phi (\, \sup_{i \geq 0} (\pi_i)^{\wedge}\, \pi_i) \qquad (\text{ by Lemma 2.8.2 }) \\ &= \phi \qquad (\text{ by Lemma 2.7.3 }). \;\clubsuit \end{aligned}$$

The projective system $\{P_i\}_{i \geq 0}$ with bonding maps $\{ e_i^j \}_{0 \leq i \leq j}$ is called the **canonical projective system** associated with P. Φ is called the **canonical isomorphism** from P_∞ onto $[\, P_\infty \to P_\infty \,]$.

It is now straightforward to show that P_∞ is a model for the lambda calculus.

Theorem 1.5: Let P be an up-complete poset with zero. Then P_∞ is an extensional environment model for the λ-cakulus.

Proof: Consider the category whose objects are the Cartesian powers of P and whose morphisms are Scott continuous functions. Then condition (1) of Theorem 1.3.1is clearly satisfied. Condition (2) follows from Lemma 2.4.3. Conditions (3) and (4a) follow from Proposition 2.6.5. Condition (4b) and extensionality follow from Theorem 3.1.4. We must show that condition (4b) holds. Let π_1 and π_2 be the projections from P_∞ to P_∞. Define $e: [P_\infty \to P_\infty] \times P_\infty \to P_\infty$ by $e(\phi, a) = \phi(a)$ for $\phi \in [P_\infty \to P_\infty]$ and $a \in P_\infty$. Let $(a, b) \in P_\infty \times P_\infty$. Then

$$\mathrm{App}(a,b) = \Phi(a)(b) = e(\Phi(a), b) = e \circ (\Phi\pi_1 \times \pi_2)(a, b).$$

Thus $\mathrm{App} = e \circ (\Phi\pi_1 \times \pi_2)$ which is Scott continuous by Proposition 2.6.4 and Lemma 2.4.3. ♣

§ 2. The Category of Ordered Models

We now define the category $\mathcal{O}$ of ordered models for the lambda calculus. The objects of $\mathcal{O}$ are up-complete posets with zero P for which there exists an order isomorphism $\Phi_P : P \to [P \to P]$. Let P and Q be ordered models. Then a morphism from P to Q is a surjective Scott continuous upper adjoint $f : P \to Q$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow \Phi_P & & \downarrow \Phi_Q \\ [P \to P] & \xrightarrow{S(f)} & [Q \to Q] \end{array}$$

Proposition 2.1: Let P and Q be up-complete posets with zero. Let $\{P_i\}_{i \geq 0}$ be the canonical projective system associated with P. Let $\{p_i^j\}_{0 \leq i \leq j}$ be the bonding maps. For $i \geq 0$ let $\pi_i : P_\infty \to P_i$ be the usual projections. Let $\{Q_i\}_{i \geq 0}$ be the canonical projective system associated with Q. Let $\{q_i^j\}_{0 \leq i \leq j}$ be the bonding maps. For each $i \geq 0$ let $\rho_i : Q_\infty \to Q_i$ be the usual projection. Let $f : P \to Q$ be a surjective Scott continuous upper adjoint. Define a sequence $\{f_i : P_i \to Q_i\}_{i \geq 0}$ as follows:

$$f_i = \begin{cases} f & \text{if } i = 0; \\ S(f_{i-1}) & \text{if } i > 0. \end{cases}$$

For each $i \geq 0$, define $F_i : P_\infty \to Q_\infty$ by $F_i = (\rho_i)^\wedge f_i \pi_i$. Then $\{F_i\}_{i \geq 0}$ is an increasing sequence, and so we may define $f_\infty : P_\infty \to Q_\infty$ by $f_\infty = \sup_{i \geq 0} F_i$. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} P_\infty & \xrightarrow{f_\infty} & Q_\infty \\ \downarrow \Phi_P & & \downarrow \Phi_Q \\ [P_\infty \to P_\infty] & \xrightarrow{S(f_\infty)} & [Q_\infty \to Q_\infty] \end{array}$$

where $\Phi_P : P_\infty \to [P_\infty \to P_\infty]$ and $\Phi_Q : Q_\infty \to [Q_\infty \to Q_\infty]$ are the canonical isomorphisms.

Proof: We show first that $f_i \geq \left(q^i_{i-1}\right)^{\wedge} f_{i-1}p^i_{i-1}$ and $(f_i)^{\wedge} \geq \left(p^i_{i-1}\right)^{\wedge} (f_{i-1})^{\wedge} q^i_{i-1}$ for $i > 0$. Let $i = 1$. Then for $\phi \in [\, P \to P \,]$ and $x \in P$,

$$\begin{aligned}
\left(\left(q^i_{i-1}\right)^{\wedge} f_{i-1}p^i_{i-1}(\phi)\right)(x) &= \left(\left(q^1_0\right)^{\wedge} f_0 p^1_0(\phi)\right)(x) \\
&= \left(\left(q^1_0\right)^{\wedge}\left(f_0\big(\phi(0)\big)\right)\right)(x) \\
&= f_0\big(\phi(0)\big) \\
&\leq f_0\Big(\phi\big((f_0)^{\wedge}(x)\big)\Big) \\
&= f_0\phi(f_0)^{\wedge}(x) \\
&= S(f_0)(\phi)(x) \\
&= f_1(\phi)(x) \\
&= f_i(\phi)(x).
\end{aligned}$$

Thus

$$\left(\left(q^i_{i-1}\right)^{\wedge} f_{i-1}p^i_{i-1}(\phi)\right)(x) \leq f_i(\phi)(x)$$

for all $x \in Q$, and so

$$\left(q^i_{i-1}\right)^{\wedge} f_{i-1}p^i_{i-1}(\phi) \leq f_i(\phi)$$

for all $\phi \in [\, P \to P \,]$. Therefore $\left(q^i_{i-1}\right)^{\wedge} f_{i-1}p^i_{i-1} \leq f_i$. Let $\psi \in [\, Q \to Q \,]$ and $y \in P$. Then

$$\begin{aligned}
\left(\left(p^i_{i-1}\right)^{\wedge} (f_{i-1})^{\wedge} q^i_{i-1}(\psi)\right)(y) &= \left(\left(p^1_0\right)^{\wedge} (f_0)^{\wedge} q^1_0(\psi)\right)(y) \\
&= \left(\left(p^1_0\right)^{\wedge}\left((f_0)^{\wedge}\big(\psi(0)\big)\right)\right)(y) \\
&= (f_0)^{\wedge}\big(\psi(0)\big) \\
&\leq (f_0)^{\wedge}\Big(\psi\big(f_0(y)\big)\Big) \\
&= (f_0)^{\wedge}\psi f_0(y) \\
&= \big(S(f_0)\big)^{\wedge}(\psi)(y) \\
&= (f_1)^{\wedge}(\psi)(y) \\
&= (f_i)^{\wedge}(\psi)(y).
\end{aligned}$$

Thus

$$\left(\left(p^i_{i-1}\right)^{\wedge} (f_{i-1})^{\wedge} q^i_{i-1}(\psi)\right)(y) \leq (f_i)^{\wedge}(\psi)(y)$$

for all $y \in P$. Therefore

$$(p_{i-1})^{\wedge} (f_{i-1})^{\wedge} q^i_{i-1}(\psi) \leq (f_i)^{\wedge}(\psi)$$

for all $\psi \in [\,Q \to Q\,]$, and so

$$(p^i_{i-1})^{\wedge}(f_{i-1})^{\wedge} q^i_{i-1} \leq (f_i)^{\wedge}.$$

We now proceed by induction. Assume that

$$(p^i_{i-1})^{\wedge}(f_{i-1})^{\wedge} q^i_{i-1} \leq (f_i)^{\wedge}$$

and

$$(q^i_{i-1})^{\wedge} f_{i-1} p^i_{i-1} \leq f_i$$

for $i > 1$. Let $\psi \in Q_{i+1}$. Then

$$\begin{aligned}
(p^{i+1}_i)^{\wedge}(f_i)^{\wedge} q^{i+1}_i(\psi) &= (p^{i+1}_i)^{\wedge}(f_i)^{\wedge} S(q^i_{i-1})(\psi) \\
&= (p^{i+1}_i)^{\wedge}(S(f_{i-1}))^{\wedge}\left(q^i_{i-1}\psi(q^i_{i-1})^{\wedge}\right) \\
&= (S(p^i_{i-1}))^{\wedge}\left((f_{i-1})^{\wedge} q^i_{i-1}\psi(q^i_{i-1})^{\wedge} f_{i-1}\right) \\
&= (p^i_{i-1})^{\wedge}(f_{i-1})^{\wedge} q^i_{i-1}\psi(q^i_{i-1})^{\wedge} f_{i-1} p^i_{i-1} \\
&\leq (f_i)^{\wedge}\psi f_i \qquad \text{(by the induction hypothesis)} \\
&= (S(f_i))^{\wedge}(\psi) \\
&= (f_{i+1})^{\wedge}(\psi).
\end{aligned}$$

Thus

$$(p^{i+1}_i)^{\wedge}(f_i)^{\wedge} q^{i+1}_i(\psi) \leq (f_{i+1})^{\wedge}(\psi)$$

for all $\psi \in Q_{i+1}$, and so

$$(p^{i+1}_i)^{\wedge}(f_i)^{\wedge} q^{i+1}_i \leq (f_{i+1})^{\wedge}.$$

Let $\phi \in P_{i+1}$. Then

$$\begin{aligned}
(q^{i+1}_i)^{\wedge} f_i p^{i+1}_i(\phi) &= (q^{i+1}_i)^{\wedge} f_i\left(S(p^i_{i-1})\right)(\phi) \\
&= (q^{i+1}_i)^{\wedge} S(f_{i-1})\left(p^i_{i-1}\phi(p^i_{i-1})^{\wedge}\right) \\
&= (S(q^i_{i-1}))^{\wedge}\left(f_{i-1}p^i_{i-1}\phi(p^i_{i-1})^{\wedge}(f_{i-1})^{\wedge}\right) \\
&= (q^i_{i-1})^{\wedge} f_{i-1}p^i_{i-1}\phi(p^i_{i-1})^{\wedge}(f_{i-1})^{\wedge} q^i_{i-1} \\
&\leq f_i\phi(f_i)^{\wedge} \qquad \text{(by the induction hypothesis)} \\
&= (S(f_i))(\phi) \\
&= f_{i+1}(\phi).
\end{aligned}$$

Thus $\left(q_i^{i+1}\right)^{\wedge} f_i p_i^{i+1}(\phi) \leq f_{i+1}(\phi)$ for all $\phi \in P_{i+1}$, and therefore $\left(q_i^{i+1}\right)^{\wedge} f_i p_i^{i+1} \leq f_{i+1}$.

We show next that $\{F_i\}_{i\geq 0}$ is an increasing sequence of functions. Now

$$\begin{aligned} F_i &= (\rho_i)^{\wedge} f_i \pi_i \\ &= \left(q_i^{i+1}\rho_{i+1}\right)^{\wedge} f_i p_i^{i+1}\pi_{i+1} \\ &= (\rho_{i+1})^{\wedge}\left(q_i^{i+1}\right)^{\wedge} f_i p_i^{i+1}\pi_{i+1} \\ &\leq (\rho_{i+1})^{\wedge} f_{i+1}\pi_{i+1} \\ &= F_{i+1}. \end{aligned}$$

We now show that $\{(F_i)^{\wedge}\}_{i\geq 0}$ is an increasing sequence. Now

$$\begin{aligned} (F_i)^{\wedge} &= (\pi_i)^{\wedge}(f_i)^{\wedge}\rho_i \\ &= \left(p_i^{i+1}\pi_{i+1}\right)^{\wedge}(f_i)^{\wedge} q_i^{i+1}\rho_{i+1} \\ &= (\pi_{i+1})^{\wedge}\left(p_i^{i+1}\right)^{\wedge}(f_i)^{\wedge} q_i^{i+1}\rho_{i+1} \\ &\leq (\pi_{i+1})^{\wedge}(f_{i+1})^{\wedge}\rho_{i+1} \\ &= (F_{i+1})^{\wedge}. \end{aligned}$$

Let I be the identity map on P_∞ and let I' be the identity map on Q_∞. Then

$$\begin{aligned} \left(\sup_{i\geq 0}(F_i)^{\wedge}\right)\circ f_\infty &= \sup_{i\geq 0}(F_i)^{\wedge}\circ F_i \qquad (\text{ by Lemma 2.8.2 }) \\ &\leq I, \end{aligned}$$

and

$$\begin{aligned} f_\infty \circ\left(\sup_{i\geq 0}(F_i)^{\wedge}\right) &= \sup_{i\geq 0} F_i \circ (F_i)^{\wedge} \qquad (\text{ by Lemma 2.8.2 }) \\ &= I'. \end{aligned}$$

Thus f_∞ is a surjective upper adjoint, and $(f_\infty)^{\wedge} = \sup_{i\geq 0}(F_i)^{\wedge}$.

Finally, we show that $\Phi_Q f_\infty = S(f_\infty)\Phi_P$. Let $x \in P_\infty$. Then

$$\begin{aligned} S(f_\infty)\circ\Phi_P(x) &= f_\infty \circ\left(\Phi_P(x)\right)\circ(f_\infty)^{\wedge} \\ &= \sup_{i\geq 0}(\rho_i)^{\wedge} f_i \pi_i (\pi_i)^{\wedge}\circ\left(\pi_{i+1}(x)\right)\circ \pi_i(\pi_i)^{\wedge}(f_i)^{\wedge}\rho_i \qquad (\text{ by Lemma 2.8.2 }) \\ &= \sup_{i\geq 0}(\rho_i)^{\wedge} f_i \circ\left(\pi_{i+1}(x)\right)\circ(f_i)^{\wedge}\rho_i \\ &= \sup_{i\geq 0}(\rho_i)^{\wedge}\circ\Big(\left(S(f_i)\right)\left(\pi_{i+1}(x)\right)\Big)\circ\rho_i \\ &= \sup_{i\geq 0}\left(S(\rho_i)\right)^{\wedge}\left(\left(f_{i+1}\pi_{i+1}(x)\right)\right). \end{aligned}$$

Thus

$$S(f_\infty) \circ \Phi_P(x) = \sup_{i \geq 0} \big(S(\rho_i) \big)^{\wedge} \circ f_{i+1} \pi_{i+1}(x)$$

for all $x \in P_\infty$. Thus

$$\begin{aligned} S(f_\infty) \circ \Phi_P &= \sup_{i \geq 0} \big(S(\rho_i) \big)^{\wedge} \circ f_{i+1} \pi_{i+1} \\ &= \sup_{i \geq 0} \big(S(\rho_i) \big)^{\wedge} \rho_{i+1} (\rho_{i+1})^{\wedge} f_{i+1} \pi_{i+1} \\ &= \Big(\sup_{i \geq 0} \big(S(\rho_i) \big)^{\wedge} \rho_{i+1} \Big) \circ \Big(\sup_{i \geq 0} (\rho_{i+1})^{\wedge} f_{i+1} \pi_{i+1} \Big) \qquad (\text{ by Lemma 2.8.2 }) \\ &= \Phi_Q f_\infty. \clubsuit \end{aligned}$$

We define a functor $\mathcal{M}$ from $\mathcal{U}$ to $\mathcal{O}$ as follows: If P is an up-complete poset with zero, then $\mathcal{M}(P) = P_\infty$. If P and Q are up-complete posets with zero and $f: P \to Q$ is a surjective Scott continuous upper adjoint, then $\mathcal{M}(f) = f_\infty$.

Lemma 2.2: Let P, Q, and R be up-complete posets with zero. Let $f: P \to Q$ and $g: Q \to R$ be surjective Scott continuous upper adjoints. Then $g_\infty f_\infty = (gf)_\infty$. Let I be the identity map on P. Then I_∞ is the identity map on P_∞.

Proof: Let $\{P_i\}_{i \geq 0}$ be the canonical projective system associated with P, with bonding maps $\{ p_i^j \}_{0 \leq i \leq j}$. Let $\{Q_i\}_{i \geq 0}$ be the canonical projective system associated with Q, with bonding maps $\{ q_i^j \}_{0 \leq i \leq j}$. Let $\{R_i\}_{i \geq 0}$ be the canonical projective system associated with Q, with bonding maps $\{ r_i^j \}_{0 \leq i \leq j}$. Define $f_i: P_i \to Q_i$ by

$$f_i = \begin{cases} f & \text{if } i = 0; \\ S(f_{i-1}) & \text{if } i > 0. \end{cases}$$

Define $g_i: Q_i \to R_i$ by

$$g_i = \begin{cases} g & \text{if } i = 0; \\ S(g_{i-1}) & \text{if } i > 0. \end{cases}$$

For each $i \geq 0$, let $\pi_i: P_\infty \to P_i$, $\rho_i: Q_\infty \to Q_i$, and $\sigma_i: R_\infty \to R_i$ be the usual projections. Then $f_\infty = \sup_{i \geq 0} (\rho_i)^{\wedge} f_i \pi_i$ and $g_\infty = \sup_{i \geq 0} (\sigma_i)^{\wedge} g_i \rho_i$. Then

$$\begin{aligned} g_\infty f_\infty &= \sup_{i \geq 0} (\sigma_i)^{\wedge} g_i \rho_i (\rho_i)^{\wedge} f_i \pi_i \qquad (\text{ by Lemma 2.8.2 }) \\ &= \sup_{i \geq 0} (\sigma_i)^{\wedge} g_i f_i \pi_i \\ &= (gf)_\infty. \end{aligned}$$

Define $I_i : P_i \to P_i$ by

$$I_i = \begin{cases} I & \text{if } i = 0; \\ S(I_{i-1}) & \text{if } i > 0. \end{cases}$$

Then I_i is the identity map on P_i for each $i \geq 0$. Thus

$$\begin{aligned} I_\infty &= \sup_{i \geq 0} (\pi_i)\hat{\ } I_i \pi_i \\ &= \sup_{i \geq 0} (\pi_i)\hat{\ } \pi_i \end{aligned}$$

which is the identity map on P_∞ by Lemma 2.7.3. ♣

We have shown that $\mathcal{M}$ is indeed a functor from **U** to **O**. We show next that $\mathcal{M}$ is a monofunctor.

§ 3. The Functor $\mathcal{M}$

We use the following notation this section: Let P be an up-complete poset with zero. Then $\Phi_P : P_\infty \to [\, P_\infty \to P_\infty \,]$ is the canonical isomorphism, and K_P is the map from P_∞ to $[\, P_\infty \to P_\infty \,]$ defined by $K_P(x)(y) = x$ for all $x, y \in P_\infty$.

The basic strategy in showing that $\mathcal{M}$ is a monofunctor is as follows. First we show that if P is an up-complete poset with zero, then P is carried isomorphically onto the set of fixed points of $\Phi_P^{-1} K_P$ by the adjoint of the projection map. Let Q be an up-complete poset with zero. Then we show that an isomorphism of $\mathcal{O}$ from P_∞ to Q_∞ carries the fixed points of $\Phi_P^{-1} K_P$ isomorphically onto the set of fixed points of $\Phi_Q^{-1} K_Q$. But then the fixed points of $\Phi_Q^{-1} K_Q$ are carried isomorphically onto Q by the projection map.

Lemma 3.1: Let P be an up-complete poset with zero. Let $\{P_i\}_{i \geq 0}$ be the canonical projective system associated with P, and let $\{P_i\}_{0 \leq i \leq j}$ be the bonding maps. Let $0 < i \leq j$. Then:

(1) $p_i^j(x) = p_{i-1}^{j-1} x \left(p_{i-1}^{j-1}\right)\hat{\ }$ for $x \in P_j$, and

(2) $\left(p_i^j\right)\hat{\ }(y) = x\left(p_{i-1}^{j-1}\right)\hat{\ }\, p_{i-1}^{j-1}$.

Proof: (1) If $i = j$ then (1) clearly holds. We proceed by induction. Assume that

$p_i^{j-1}(x) = p_{i-1}^{j-2}x(p_{i-1}^{j-2})\hat{}$ for $x \in P_{j-1}$ and $j > i$. Then

$$\begin{aligned} p_i^j(x) &= p_i^{j-1}p_{j-1}^j(x) \\ &= p_i^{j-1}(S(p_{j-2}^{j-1})(x)) \\ &= p_i^{j-1}\left(p_{j-2}^{j-1}x(p_{j-2}^{j-1})\hat{}\right) \\ &= p_{i-1}^{j-2}p_{j-2}^{j-1}x(p_{j-2}^{j-1})\hat{}\,(p_{i-1}^{j-2})\hat{} \qquad (\text{ by the induction hypothesis }) \\ &= p_{i-1}^{j-1}x(p_{i-1}^{j-2}p_{j-2}^{j-1})\hat{} \\ &= p_{i-1}^{j-2}x(p_{i-1}^{j-1})\hat{}\,. \end{aligned}$$

(2) If $i = j$ then (2) clearly holds, and we again proceed by induction. Assume that $(p_i^{j-1})\hat{}\,(z) = (p_{i-1}^{j-2})\hat{}\,zp_{i-1}^{j-2}$ for $z \in P_i$ and $j > i$. Then

$$\begin{aligned} (p_i^j)\hat{}\,(z) &= (p_i^{j-1}p_{j-1}^j)\hat{}\,(y) \\ &= (p_{j-1}^j)\hat{}\,(p_i^{j-1})\hat{}\,(y) \\ &= \left(S(p_{j-2}^{j-1})\right)\hat{}\left((p_i^{j-1})\hat{}\,(y)\right) \\ &= (p_{j-2}^{j-1})\hat{} \circ \left((p_i^{j-1})\hat{}\,(y)\right) \circ p_{j-2}^{j-1} \\ &= (p_{j-2}^{j-1})\hat{}\,(p_{i-1}^{j-2})\hat{}\,yp_{i-1}^{j-2}p_{j-2}^{j-1} \qquad (\text{ by the induction hypothesis }) \\ &= (p_{i-1}^{j-2}p_{j-2}^{j-1})\hat{}\,yp_{i-1}^{j-1} \\ &= (p_{i-1}^{j-1})\hat{}\,yp_{i-1}^{j-1}. \clubsuit \end{aligned}$$

Lemma 3.2: Let P be an up-complete poset with zero, and let $\{P_i\}_{i \geq 0}$ be the canonical projective system associated with P. Let $\{P_i^j\}_{0 \leq i \leq j}$ be the bonding maps. For each $i \geq 0$ define $k_i : P_i \to P_{i+1}$ by $k_i(x)(y) = x$ for $x, y \in P_i$. Let I be the identity map on P. Then:
(1) $p_i^{i+1}k_i = k_{i-1}p_{i-1}^i$ for $i \geq 1$, and
(2) $k_i \ldots k_0 = (p_0^{i+1})\hat{}$ for $i \geq 0$.

Proof: (1) Let $x \in P_i$ and $y \in P_{i-1}$. Then

$$\begin{aligned}(p_i^{i+1}k_i(x))(y) &= \left(S(p_{i-1}^i)\right) \circ k_i(x)(y)\\ &= p_{i-1}^i \circ (k_i(x)) \circ (p_{i-1}^i)^\wedge(y)\\ &= p_{i-1}^i(x)\\ &= k_{i-1}(p_{i-1}^i(x))(y).\end{aligned}$$

Thus

$$p_i^{i+1}k_i(x)(y) = k_{i-1}p_{i-1}^i(x)(y)$$

for all $y \in P_{i-1}$, and therefore

$$p_i^{i+1}k_i(x) = k_{i-1}p_{i-1}^i(x)$$

for all $x \in P_i$. Thus $p_i^{i+1}k_i = k_{i-1}p_{i-1}^i$.

(2) If $i = 0$ then (2) clearly holds. Let $i = 1$. Let $x \in P_0$ and $y \in P_1$. Then

$$\begin{aligned}(p_0^{i+1})^\wedge(x)(y) &= (p_0^2)^\wedge(x)(y)\\ &= (p_0^1p_1^2)^\wedge(x)(y)\\ &= (p_1^2)^\wedge\left((p_0^1)^\wedge(x)\right)(y)\\ &= \left(S(p_0^1)\right)^\wedge \circ \left((p_0^1)^\wedge(x)\right)(y)\\ &= (p_0^1)^\wedge \circ \left((p_0^1)^\wedge(x)\right) \circ p_0^1(y)\\ &= (p_0^1)^\wedge(x)\\ &= k_0(x)\\ &= k_1(k_0(x))(y)\\ &= k_i \dots k_0(x)(y).\end{aligned}$$

Thus

$$k_i \dots k_0(x)(y) = (p_0^{i+1})^\wedge(x)(y)$$

for all $y \in P_i$, and therefore

$$k_i \dots k_0(x) = (p_0^{i+1})^\wedge(x)$$

for all $x \in P_{i-1}$. Thus $k_i \dots k_0 = (p_0^{i+1})^\wedge$. We now proceed by induction. Let $i \geq 2$, and assume that $k_j \dots k_0 = (p_0^{j+1})\hat{}$ for all $j < i$. Let $x \in P_0$ and $y \in P_i$. Then

$$\begin{aligned}
(p_0^{i+1})\hat{}\,(x)(y) &= (p_0^i p_i^{i+1})\hat{}\,(x)(y) \\
&= (p_i^{i+1})\hat{}\,(p_0^i)\hat{}\,(x)(y) \\
&= \big(S(p_{i-1}^i)\big)\hat{} \circ (k_{i-1} \dots k_0)(x)(y) \qquad (\text{ by the induction hypothesis }) \\
&= (p_{i-1}^i)\hat{} \circ (k_{i-1} \dots k_0(x)) \circ p_i^{i-1}(y) \\
&= (p_{i-1}^i)\hat{} \circ \big(k_{i-1}(k_{i-2} \dots k_0(x))\big) \circ p_i^{i-1}(y) \\
&= (p_{i-1}^i)\hat{}\,(k_{i-2} \dots k_0(x)) \\
&= (p_{i-1}^i)\hat{} \circ (p_0^{i-1})\hat{}\,(x) \qquad (\text{ by the induction hypothesis }) \\
&= (p_0^{i-1} p_{i-1}^i)\hat{}\,(x) \\
&= (p_0^i)\hat{}\,(x) \\
&= k_{i-1} \dots k_0(x) \qquad (\text{ by the induction hypothesis }) \\
&= k_i(k_{i-1} \dots k_0(x))(y) \\
&= k_i \dots k_0(x)(y).
\end{aligned}$$

Thus

$$(p_0^{i+1})\hat{}\,(x)(y) = k_i \dots k_0(x)(y)$$

for all $y \in P_i$, and so $(p_0^{i+1})\hat{}\,(x) = k_i \dots k_o(x)$ for all $x \in P_0$. Therefore $(p_0^{i+1})\hat{} = k_i \dots k_0$. ♣

Lemma 3.8: Let P be an up-complete poset with zero. Let $\{P_i\}_{i \geq 0}$ be the canonical projective system associated with P, with bonding maps $\{p_i^j\}_{0 \leq i \leq j}$. For each $i \geq 0$, let $\pi_i : P_\infty \to P_i$ be the usual projection. For $i \geq 0$, define $k_i : P_i \to P_{i+1}$ by $k_i(x)(y) = x$ for $x, y \in P_i$.

(1) Define $k : P_\infty \to P_\infty$ by

$$\pi_i k(x) = \begin{cases} \pi_0(x) & \text{if } i = 0; \\ k_{i-1}\pi_{i-1}(x) & \text{if } i > 0 \end{cases}$$

for $x \in P_\infty$. Then $k = \Phi_P^{-1} K_P$.

(2) For $i \geq 0$, define $k^i : P_\infty \to P_\infty$ by

$$k^i = \begin{cases} I & \text{if } i = 0; \\ k \circ k^{i-1} & \text{if } i > 0. \end{cases}$$

Thus k^i is the i^{th} iterate of k. Then $\pi_i k^i = k_{i-1} \dots k_0(\pi_0(x))$.

(3) An element x of P_∞ is a fixed point of $\Phi_P^{-1} K_P$ if and only if $x \in (\pi_0)\hat{}\,(P)$.

Proof: (1) We will show first that $k(x) \in P_\infty$ when $x \in P_\infty$. Let $i = 1$. Then

$$\begin{aligned} p^i_{i-1}\pi_i k(x) &= p^1_0 \pi_1 k(x) \\ &= k_0(\pi_0(x))(0) \\ &= \pi_0(x) \\ &= \pi_0 k(x) \\ &= \pi_{i-1} k(x). \end{aligned}$$

Let $i > 1$. Then

$$\begin{aligned} p^i_{i-1}\pi_i k(x) &= p^i_{i-1} k_{i-1} \pi_{i-1}(x) \\ &= k_{i-2} p^{i-1}_{i-2} \pi_{i-1}(x) \qquad (\text{ by Lemma 3.3.2 }) \\ &= k_{i-2}\pi_{i-2}(x) \\ &= \pi_{i-1} k(x). \end{aligned}$$

Thus $p^i_{i-1}\pi_i k(x) = \pi_{i-1} k(x)$ for all $i > 0$, and so $k(x) \in P_\infty$.

Now we compute $\Phi_P k$. Let $x, y \in P_\infty$. Then

$$\begin{aligned} \Phi_P k(x)(y) &= \sup_{i \geq 0} (\pi_i)\hat{} \circ \Big(\pi_{i+1}(k(x))\Big) \circ \pi_i(y) \\ &= \sup_{i \geq 0} (\pi_i)\hat{} \circ \big(k_i \pi_i(x)\big) \circ \pi_i(y) \\ &= \sup_{i \geq 0} (\pi_i)\hat{}\,\big(\pi_i(x)\big) \\ &= x \qquad (\text{ by Lemma 2.7.3 }) \\ &= K_P(x)(y). \end{aligned}$$

Thus $\Phi_P k(x)(y) = K_P(x)(y)$ for all $y \in P_\infty$, and so $\Phi_P k(x) = K_P(x)$ for all $x \in P_\infty$. Therefore $\Phi_P k = K_P$, and thus $k = \Phi_P^{-1} K_P$.

(2) Statement (2) clearly holds for $i = 0$. Let $i = 1$. Then

$$\begin{aligned} \pi_i k^i(x) &= \pi_1 k(x) \\ &= k_0(\pi_0(x)). \end{aligned}$$

We proceed by induction. Let $i > 0$ and assume that

$$\pi_{i-1} k^{i-1}(x) = k_{i-2} \dots k_0(\pi_0(x)).$$

Then

$$\begin{aligned}
\pi_i k^i(x) &= \pi_i k k^{i-1}(x) \\
&= k_{i-1}\pi_{i-1}k^{i-1}(x) \\
&= k_{i-1}k_{i-2}\cdots k_0(\pi_0(x)) \qquad (\text{ by the induction hypothesis }) \\
&= k_{i-1}\cdots k_0(\pi_0(x)).
\end{aligned}$$

(3) Let x be a fixed point of $\Phi_P^{-1}K_P$. Then $k^i(x) = x$ for all $i \geq 0$. Thus

$$\begin{aligned}
\pi_i(x) &= \pi_i k^i(x) \\
&= k_{i-1}\cdots k_0\pi_0(x) \\
&= (p_0^i)\hat{\ }\pi_0(x) \\
&= \pi_i(\pi_0)\hat{\ }\pi_0(x).
\end{aligned}$$

Thus $\pi_i(x) = \pi_i(\pi_0)\hat{\ }\pi_0(x)$ for all $i \geq 0$, and therefore $x = (\pi_0)\hat{\ }\pi_0(x)$. We have shown that $x \in (\pi_0)\hat{\ }(P)$.

Conversely, let $x \in (\pi_0)\hat{\ }(P)$. Then $x = (\pi_0)\hat{\ }\pi_0(x)$. Let $y \in P_\infty$. Then

$$\begin{aligned}
\Phi_P(x)(y) &= \sup_{i\geq 0}(\pi_i)\hat{\ }\circ\left(\pi_{i+1}(\pi_0)\hat{\ }\pi_0(x)\right)\circ\pi_i(y) \\
&= \sup_{i\geq 0}(\pi_i)\hat{\ }\circ\left((p_0^{i+1})\hat{\ }\pi_0(x)\right)\circ\pi_i(y) \\
&= \sup_{i\geq 0}(\pi_i)\hat{\ }\circ\left((p_0^1 p_1^{i+1})\hat{\ }\pi_0(x)\right)\circ\pi_i(y) \\
&= \sup_{i\geq 0}(\pi_i)\hat{\ }\circ\left((p_1^{i+1})\hat{\ }(p_0^1)\hat{\ }\pi_0(x)\right)\circ\pi_i(y) \\
&= \sup_{i\geq 0}(\pi_i)\hat{\ }\circ(p_0^i)\hat{\ }\circ\left((p_0^1)\hat{\ }\pi_0(x)\right)\circ p_0^i\pi_i(x) \\
&= \sup_{i\geq 0}(\pi_i)\hat{\ }(p_0^i)\hat{\ }(\pi_0(x)) \\
&= \sup_{i\geq 0}(p_0^i\pi_i)\hat{\ }\pi_0(x) \\
&= \sup_{i\geq 0}(\pi_0)\hat{\ }\pi_0(x) \\
&= x \\
&= K_P(x)(y).
\end{aligned}$$

Thus $\Phi_P(x)(y) = K_P(x)(y)$ for all $y \in P_\infty$, and therefore $\Phi_P(x) = K_P(x)$. It follows that x is a fixed point of $\Phi_P^{-1}K_P$. ♣

Proposition 8.4: Let P and Q be up-complete posets with zero. Let $f:P \to Q$ be a surjective Scott continuous upper adjoint. Then $\hat{f}$ is an order isomorphism onto $\hat{f}(P)$.

Proof: Let $\tilde{f}$ be the restriction of f to $\hat{f}(P)$. Let $x \in P$. Then $\tilde{f}\hat{f}(x) = f\hat{f}(x) = x$. Let $y \in \hat{f}(P)$. Then $y = \hat{f}(z)$ for some $z \in P$. Thus

$$\begin{aligned}\hat{f}\tilde{f}(y) &= \hat{f}\tilde{f}\hat{f}(z)\\ &= \hat{f}f\hat{f}(z)\\ &= \hat{f}(z) = y.\end{aligned}$$

Thus $\tilde{f} = \hat{f}^{-1}$ and so $\hat{f}$ is an order isomorphism. ♣

Theorem 8.5: Let P and Q be up-complete posets with zero. If P_∞ is isomorphic to Q_∞ in the category $\mathcal{O}$, then P is order isomorphic to Q. Thus $\mathcal{M}$ is a monofunctor from $\mathcal{U}$ to $\mathcal{O}$.

Proof: Let $\phi: P_\infty \to Q_\infty$ be the isomorphism. Then $\Phi_Q\phi = S(\phi)\Phi_P$. Let $\pi_0: P_\infty \to P$ and $\rho_0: Q_\infty \to Q$ be the usual projections. Now $(\pi_0)^\wedge(P)$ is the set of fixed points of $\Phi_P^{-1}K_P$ and $(\rho_0)^\wedge(Q)$ is the set of fixed points of $\Phi_Q^{-1}K_Q$. We show that

$$\phi\big((\pi_0)^\wedge(P)\big) \subseteq (\rho_0)^\wedge(Q).$$

Let $x \in P_\infty$ be a fixed point of $\Phi_P^{-1}K_P$. Then $\Phi_P(x) = K_P(x)$. Let $y \in Q_\infty$. Then

$$\begin{aligned}\Phi_Q\big(\phi(x)\big)(y) &= \big(S(\phi)\big) \circ \big(\Phi_P(x)\big)(y)\\ &= \phi \circ \big(\Phi_P(x)\big) \circ \phi^{-1}(y)\\ &= \phi \circ \big(K_P(x)\big) \circ \phi^{-1}(y)\\ &= \phi(x)\\ &= K_Q\big(\phi(x)\big)(y).\end{aligned}$$

Thus

$$\Phi_Q\big(\phi(x)\big)(y) = K_Q\big(\phi(x)\big)(y)$$

for all $y \in Q_\infty$, and so $\Phi_Q(\phi(x)) = K_Q(\phi(x))$. Thus $\phi(x)$ is a fixed point of $\Phi_Q^{-1}K_Q$.

Dually,

$$\phi^{-1}\big((\rho_0)^\wedge(Q)\big) \subseteq (\pi_0)^\wedge(P).$$

Thus the restriction of ϕ to $(\pi_0)\hat{}\,(P)$ is an order isomorphism onto $(\rho_0)\hat{}\,(Q)$. Therefore, by Proposition 3.3.4, $\rho_0\phi(\pi_0)\hat{}$ is an order isomorphism of P onto Q. ♣

§ 4. Values of Combinators

In this section we calculate the values of the combinators **I**, **K**, and **Y** in the canonical models. Througout this section we use the following notation: The canonical projective system associated with P is $\{P_i\}_i \geq 0$. The bonding maps are $\{p_i^j\}_{0 \leq i \leq j}$. The zero of P is 0. For each $i \geq 0$, the identity map on P_i is I_i. The identity map on P_∞ is Id. The usual projection maps on P_∞ are $\{\pi_i\}_{i \geq 0}$. The set of variables for the lambda calculus is V. The canonical isomorphism from P_∞ to $[\, P_\infty \rightarrow P_\infty\,]$ is Φ. We define $K: P_\infty \rightarrow P_\infty$ by $K(a)(b) = a$ for all $a, b \in P_\infty$.

First we calculate $[\![\mathbf{I}]\!]$. We define an element $I \in \prod_{i \in I} P_i$ as follows:

$$\pi_i(I) = \begin{cases} 0 & \text{if } i = 0; \\ I_{i-1} & \text{if } i > 0. \end{cases}$$

Then

$$\begin{aligned} p_0^1\pi_1(I) &= \pi_1(I)(0) \\ &= I_0(0) \\ &= 0 \\ &= \pi_0(I). \end{aligned}$$

For $i > 0$,

$$\begin{aligned} p_i^{i+1}(I) &= p_{i-1}^i \circ (\pi_{i+1}(I)) \circ (p_{i-1}^i)\hat{} \\ &= p_{i-1}^i I_i (p_{i-1}^i)\hat{} \\ &= p_{i-1}^i (p_{i-1}^i)\hat{} \\ &= I_{i-1} \\ &= \pi_i(I). \end{aligned}$$

Thus $I \in P_\infty$.

We will show that $[\![\mathbf{I}]\!] = I$. Let $\rho: V \rightarrow P_\infty$ be an environment. Then

$$[\![\mathbf{I}]\!] = [\![\lambda x.x]\!] = \Phi^{-1}(f)$$

where $f: P_\infty \to P_\infty$ is defined by $f(a) = [x]\left\{ \frac{d}{x} \right\} = d$. Thus $f = \mathrm{Id}$. Therefore

$$\begin{aligned}
\Phi^{-1}(f) &= \sup_{i \geq 0} (\pi_{i+1})\hat{}\,(\pi_i \mathrm{Id}(\pi_i)\hat{}\,) \\
&= \sup_{i \geq 0} (\pi_{i+1})\hat{}\, I_i \\
&= \sup_{i \geq 0} (\pi_{i+1})\hat{} \circ \pi_{i+1}(I) \\
&= I \qquad (\text{ by Lemma 2.7.3 }).
\end{aligned}$$

Thus $[\mathbf{I}]\rho = I$ for every environment ρ.

Next we compute $[\mathbf{K}]$. For each $i \geq 0$, define $k_i: P_i \to P_{i+1}$ by $k_i(x)(y) = x$ for $x, y \in P_i$. Define $k \in \prod_{i \in I} P_i$ by

$$\pi_i(k) = \begin{cases} 0 & \text{if } i = 0; \\ p^i_{i-1} k_{i-1} & \text{if } i > 0. \end{cases}$$

Then

$$\begin{aligned}
p^1_0 \pi_1(k) &= \pi_1(k)(0) \\
&= p^1_0 k_0(0) \\
&= k_0(0)(0) \\
&= 0.
\end{aligned}$$

For $i > 0$,

$$\begin{aligned}
p^{i+1}_i(\pi_{i+1}(k)) &= p^{i+1}_i(p^{i+1}_i k_i) \\
&= p^i_{i-1} p^{i+1}_i k_i (p^i_{i-1})\hat{} \\
&= p^i_{i-1} k_{i-1} p^i_{i-1} (p^i_{i-1})\hat{} \qquad (\text{ by Lemma 3.3.2 }) \\
&= p^i_{i-1} k_{i-1} \\
&= \pi_i(k).
\end{aligned}$$

Thus $k \in P_\infty$. We show that $[\mathbf{K}] = k$. Let $\rho: V \to P_\infty$ be an environment. Then

$$[\mathbf{K}]\rho = [\lambda x \lambda y.x]\rho = \Phi^{-1}(f),$$

where $f: P_\infty \to P_\infty$ is defined by

$$f(a) = [\lambda y.x]\rho\left\{ \frac{a}{x} \right\}$$

for $a \in P_\infty$. Thus $f(a) = \Phi^{-1}(g)$, where $g: P_\infty \to P_\infty$ is defined by

$$g(b) = [x]\rho\left\{\frac{a}{x}\right\}\left\{\frac{b}{y}\right\}$$
$$= a$$
$$= K(a)(b)$$

for $b \in P_\infty$. Thus $g = K(a)$, and so $f(a) = \Phi^{-1}(K(a))$ for $a \in P_\infty$. Therefore $f = \Phi^{-1}K$, and hence

$$
\begin{aligned}
[\mathbf{K}]\rho &= \Phi^{-1}(\Phi^{-1}K) \\
&= \sup_{i \geq 0} (\pi_{i+1})\hat{}\left(\pi_i \circ (\Phi^{-1}K) \circ (\pi_i)\hat{}\right) \\
&= \sup_{i \geq 0} (\pi_{i+1})\hat{}\left(k_{i-1}\pi_{i-1}(\pi_i)\hat{}\right) && \text{(by Lemma 3.3.3)} \\
&= \sup_{i \geq 0} (\pi_{i+1})\hat{}\left(k_{i-1}p^i_{i-1}\right) \\
&= \sup_{i \geq 0} (\pi_{i+1})\hat{}\left(p^{i+1}_i k_i\right) && \text{(by Lemma 3.3.2)} \\
&= \sup_{i \geq 0} (\pi_{i+1})\hat{}\left(\pi_{i+1}(k)\right) \\
&= k && \text{(by Lemma 2.7.3).}
\end{aligned}
$$

Before we calculate $[\mathbf{Y}]$, we need a few more facts about up-complete posets with zero.

Lemma 4.1: Let P be an up-complete poset with zero. Let $f: P \to [P \to P]$ be a Scott continuous map. Define $g: P \to P$ by $g(x) = f(x)(x)$. Then g is Scott continuous.

Proof: Let D be a directed set in P. Then

$$\begin{aligned} g(\sup D) &= f(\sup D)(\sup D) \\ &= f(\sup D)\Big(\sup_{d\in D} d\Big) \\ &= \sup_{d\in D} f(\sup D)(d) \\ &= \sup_{d\in D} f\Big(\sup_{e\in D} e\Big)(d) \\ &= \sup_{d\in D}\Big(\sup_{e\in D} f(e)\Big)(d) \\ &= \sup_{d\in D}\Big(\sup_{e\in D} f(e)(d)\Big) \\ &= \sup_{d\in D} f(d)(d) \qquad (\text{ by Lemma 2.8.1 }) \\ &= \sup_{d\in D} g(d) \\ &= \sup g(D). \clubsuit \end{aligned}$$

We prove next that Scott continuous maps on up-complete posets with zero always have least fixed points.

Proposition 4.2: Let P be an up-complete poset with zero. Let D be the zero of P. Let $f, g_1, g_2, \ldots, g_n \in [\, P \to P\,]$, with $g_1 \leq g_2 \leq \ldots \leq g_n \leq f$. Define a sequence $\{a_i\}_{i\geq 0}$ by

$$a_i = \begin{cases} 0 & \text{if } i = 0; \\ g_i(a_{i-1}) & \text{if } 1 \leq i \leq n; \\ f(a_{i-1}) & \text{if } i > n. \end{cases}$$

Then the sequence is increasing, and $\sup_{i\geq 0} a_i$ is the least fixed point of f.

Proof: First we show that the sequence $\{a_i\}_{i\geq 0}$ is increasing. Now $a_0 = 0 \leq a_i$. We proceed by induction. Let $0 < i < n$ and assume that $a_{i-1} \leq a_i$. Then

$$a_i = g_i(a_{i-1}) \leq g_i(a_i) \leq g_{i+1}(a_i) = a_{i+1}.$$

Thus $a_i \leq a_{i+1}$. We have shown that $a_i \leq a_{i+1}$ for $0 \leq i \leq n$. Now

$$a_n = g_n(a_{n-1}) \leq f(a_{n-1}) \leq f(a_n) = a_{n+1}.$$

Again we proceed by induction. Let $i > n$, and assume that $a_{i-1} \leq a_i$. Then

$$a_i = f(a_{i-1}) \leq f(a_i) = a_{i+1}.$$

Thus $a_i \le a_{i+1}$ for all $i \ge 0$. Furthermore,

$$\begin{aligned} f\Big(\sup_{i\ge 0} a_i\Big) &= f\Big(\sup_{i>n} a_i\Big) \\ &= \sup_{i>n} f(a_i) \\ &= \sup_{i>n} a_{i+1} \\ &= \sup_{i\ge 0} a_i. \end{aligned}$$

Thus $\sup_{i\ge 0} a_i$ is a fixed point of f. For $i \ge 0$ let f^i be the i^{th} iterate of f. We show that $a_i \le f^i(0)$ for $i \ge 0$. Clearly $a_0 \le f^0(0)$. We proceed by induction. Let $i > 0$, and assume that $a_{i-1} \le f^{i-1}(0)$. If $i \le n$ then

$$\begin{aligned} a_i &= g_i(a_{i-1}) \\ &\le g_i\big(f^{i-1}(0)\big) && \text{(by the induction hypothesis)} \\ &\le f\big(f^{i-1}(0)\big) \\ &= f^i(0). \end{aligned}$$

If $i > n$ then

$$\begin{aligned} a_i &= f(a_{i-1}) \\ &\le f(f^{i-1}(0)) && \text{(by the induction hypothesis)} \\ &= f^i(0). \end{aligned}$$

Thus $a_i \le f^i(0)$ for all $i \ge 0$. Let y be a fixed point of f. Then $f^i(y) = y$ for all $i \ge 0$. Thus

$$\begin{aligned} \sup_{i\ge 0} a_i &\le \sup_{i\ge 0} f^i(0) \\ &\le \sup_{i\ge 0} y \\ &= y. \end{aligned}$$

Therefore $\sup_{i\ge 0} a_i$ is the least fixed point of f. ♣

Note that this proposition holds in the case $n = 0$; i.e., $a_i = f^i(0)$. In this case we have the fixed point theorem of Tarski [Gierz, et. al., 1980, pp. 8,9]. This generalization of the Tarski fixed point theorem is needed for the calculation of **[Y]**.

Proposition 4.8: Let P be an up-complete poset with zero. Let 0 be the zero of P. Define $F\colon [\, P \to P\,] \to P$ by $F(\phi) = \sup_{i\ge 0} \phi^i(0)$. Thus $F(\phi)$ is the least fixed point of ϕ. Then F is Scott continuous.

Proof: Let D be a directed set in $[\,P \to P\,]$. Then

$$\begin{aligned}
F(\sup D) &= \sup_{i\geq 0}(\sup D)^{i}(0) \\
&= \sup_{i\geq 0}(\sup_{d\in D} d)^{i}(0) \\
&= \sup_{i\geq 0}\Big(\sup_{d\in D} d^{i}\Big)(0) \qquad (\text{ by Lemma 2.8.2 }) \\
&= \sup_{i\geq 0}\Big(\sup_{d\in D} d^{i}(0)\Big) \\
&= \sup_{d\in D}\Big(\sup_{i\geq 0} d^{i}(0)\Big) \qquad (\text{ Lemma 2.8.1 }) \\
&= \sup_{d\in D} F(d) \\
&= \sup F(D). \clubsuit
\end{aligned}$$

We now calculate $[\mathbf{Y}]$. For $i \geq 0$ let 0_i be the zero of P_i. Define $F \in \prod_{i\in I} P_i$ by

$$\pi_i(F) = \begin{cases} 0 & \text{if } i = 0; \\ I_0 & \text{if } i = 1; \\ F_i & \text{if } i > 1; \end{cases}$$

where $F_i : P_{i-1} \to P_{i-1}$ is defined by

$$F_i(\phi) = \left(p_{i-2}^{i-1}\right)^{\wedge}\Big(\sup_{j\geq 0} \phi^{j}(0_{i-2})\Big).$$

Note that each F_i is continuous by Proposition 3.4.3. Now

$$\begin{aligned}
p_0^1 \pi_1(F) &= \pi_1(F)(0) \\
&= I_0(0) \\
&= 0 \\
&= \pi_0(F).
\end{aligned}$$

Let $x \in P$. Then

$$\begin{aligned}
p_1^2 \pi_2(F)(x) &= p_0^1 \circ \left(\pi_2(F)\right) \circ \left(p_0^1\right)^{\wedge}(x) \\
&= p_0^1 \left(p_0^1\right)^{\wedge}\Big(\sup_{j\geq 0}\Big(\left(p_0^1\right)^{\wedge}(x)\Big)^{j}(0)\Big) \\
&= \sup_{j\geq 0} x \\
&= x.
\end{aligned}$$

Thus $p_1^2\pi_2(F) = I_0 = \pi_1(F)$. Now let $i \geq 2$. Let $x \in P_{i-1}$. Then

$$\begin{aligned}
p_i^{i+1}\pi_{i+1}(F)(x) &= p_{i-1}^i \circ \left(\pi_{i+1}(F)\right) \circ \left(p_{i-1}^i\right)^{\wedge}(x) \\
&= p_{i-1}^i\left(p_{i-1}^i\right)^{\wedge}\left(\sup_{j\geq 0}\left(\left(p_{i-1}^i\right)^{\wedge}(x)\right)^j(0_{i-1})\right) \\
&= \sup_{j\geq 0}\left(\left(p_{i-2}^{i-1}\right)^{\wedge} x p_{i-2}^{i-1}\right)^j(0_{i-1}) \\
&= \sup_{j\geq 0}\left(p_{i-2}^{i-1}\right)^{\wedge} x^j p_{i-2}^{i-1}(0_{i-1}) \\
&= \left(p_{i-2}^{i-1}\right)^{\wedge}\left(\sup_{j\geq 0} x^j(0_{i-2})\right) \\
&= F_i(x) \\
&= \pi_i(F)(x).
\end{aligned}$$

Thus $p_i^{i+1}\pi_{i+1}(F) = \pi_i(F)$ for $i \geq 0$, and therefore $F \in P_\infty$.

We show that $[\![Y]\!] = F$. Let $\rho: V \to P_\infty$ be an environment. Then

$$[\![\mathbf{Y}]\!]\rho = [\lambda y.(\lambda x.y(xx))(\lambda x.y(xx))]\,\rho = \Phi^{-1}(h),$$

where $h: P_\infty \to P_\infty$ is defined by

$$h(a) = \Phi\left([\![\lambda x.y(xx)]\!]\rho\left\{\frac{a}{y}\right\}\right)\left([\![\lambda x.y(xx)]\!]\rho\left\{\frac{a}{y}\right\}\right)$$

for $a \in P_\infty$. Thus we must calculate $[\![\lambda x.y(xx)]\!]\rho\left\{\frac{a}{y}\right\}$ for $x \neq y$.

We define $g: P_\infty \to \prod_{i\in I} P_i$ as follows. Let $x \in P_\infty$. Then

$$\pi_i(g)(x) = \begin{cases} \pi_0(x) & \text{if } i = 0; \\ g_{i,x} & \text{if } i > 0; \end{cases}$$

where $g_{i,x}: P_{i-1} \to P_{i-1}$ is defined by $g_{i,x}(y) = \left(\pi_i(x)\right) \circ \left(\left(p_{i-1}^i\right)^{\wedge}(y)\right)(y)$ for $y \in P_{i-1}$. Now each $g_{i,x}$ is Scott continuous by Lemma 3.4.1. We now show that $g: P_\infty \to P_\infty$. Let

$x \in P_\infty$. Then

$$
\begin{aligned}
p_0^1 \pi_i(g)(x) &= \pi_1(g)(x)(0) \\
&= g_{1,x}(0) \\
&= \left(\pi_1(x)\right) \circ \left(\left(p_0^1\right)^{\wedge}(0)\right)(0) \\
&= \pi_1(x)(0) \\
&= p_0^1(x)(0) \\
&= \pi_0(x) \\
&= \pi_0(g)(x).
\end{aligned}
$$

Let $i > 0$ and $y \in P_{i-1}$. Then

$$
\begin{aligned}
\left(p_i^{i+1} \pi_{i+1}(g)(x)\right)(y) &= p_{i-1}^i \circ \left(\pi_{i+1}(g)(x)\right) \circ \left(p_{i-1}^i\right)^{\wedge}(y) \\
&= p_{i-1}^i g_{i+1,x}\left(\left(p_{i-1}^i\right)^{\wedge}(y)\right) \\
&= p_{i-1}^i \circ \left(\pi_{i+1}(x)\right) \circ \left(\left(p_i^{i+1}\right)^{\wedge}\left(\left(p_{i-1}^i\right)^{\wedge}(y)\right)\right)\left(\left(p_{i-1}^i\right)^{\wedge}(y)\right) \\
&= p_{i-1}^i \circ \left(\pi_{i+1}(x)\right) \circ \left(\left(p_{i-1}^{i+1}\right)^{\wedge}(y)\right) \circ \left(p_{i-1}^i\right)^{\wedge}(y) \\
&= p_{i-1}^i \circ \left(\pi_{i+1}(x)\right) \circ \left(p_{i-2}^i\right)^{\wedge} y p_{i-2}^i \left(p_{i-1}^i\right)^{\wedge}(y) \\
&= p_{i-1}^i \circ \left(\pi_{i+1}(x)\right) \circ \left(p_{i-1}^i\right)^{\wedge}\left(p_{i-2}^{i-1}\right)^{\wedge} y p_{i-2}^{i-1} p_{i-1}^i \left(p_{i-1}^i\right)^{\wedge}(y) \\
&= \left(p_i^{i+1} \pi_{i+1}(x)\right) \circ \left(\left(p_{i-1}^i\right)^{\wedge}(y)\right)(y) \\
&= \left(\pi_i(x)\right) \circ \left(\left(p_{i-1}^i\right)^{\wedge}(y)\right)(y) \\
&= g_{i,x}(y) \\
&= \pi_i(g)(x)(y).
\end{aligned}
$$

Thus $p_i^{i+1} \pi_{i+1}(g)(x) = \pi_i(g)(x)$ for all $i \geq 0$, and so $g(x) \in P_\infty$.

We show next that $[\lambda x. y(xx)]\rho\left\{ \frac{a}{y} \right\} = g(x)$ for all $a \in P_\infty$. Let $a \in P_\infty$. Then

$[\![\lambda x.y(xx)]\!]\rho\left\{\frac{a}{y}\right\} = \Phi^{-1}(f)$, where $f: P_\infty \to P_\infty$ is defined by

$$\begin{aligned}
f(b) &= [\![y(xx)]\!]\rho\left\{\frac{a}{y}\right\}\left\{\frac{b}{x}\right\}\\
&= \Phi\left([\![y]\!]\rho\left\{\frac{a}{y}\right\}\left\{\frac{b}{x}\right\}\right)\left([\![xx]\!]\rho\left\{\frac{a}{y}\right\}\left\{\frac{b}{x}\right\}\right)\\
&= \Phi(a)\left(\Phi\left([\![x]\!]\rho\left\{\frac{a}{y}\right\}\left\{\frac{b}{x}\right\}\right)\left([\![x]\!]\rho\left\{\frac{a}{y}\right\}\left\{\frac{b}{x}\right\}\right)\right)\\
&= \Phi(a)\big(\Phi(b)(b)\big).
\end{aligned}$$

Thus $\Phi^{-1}(f) = \sup_{i\geq 0}\left(\pi_i f(\pi_i)^{\wedge}\right)$. Let $y \in P_i$. Then

$$\begin{aligned}
\pi_i f(\pi_i)^{\wedge}(y) &= \pi_i\Big(\Phi(a)\Big(\Phi\big((\pi_i)^{\wedge}(y)\big)\big((\pi_i)^{\wedge}(y)\big)\Big)\Big)\\
&= \pi_i \circ \big(\Phi(a)\big) \circ \Big(\Phi\big((\pi_i)^{\wedge}(y)\big)\Big)\big((\pi_i)^{\wedge}(y)\big)\\
&= \sup_{j\geq 0} \pi_i(\pi_j)^{\wedge} \circ \big(\pi_{j+1}(a)\big) \circ \pi_j(\pi_j)^{\wedge} \circ \Big(\pi_{j+1}\big((\pi_i)^{\wedge}(y)\big)\Big) \circ \pi_j(\pi_i)^{\wedge}(y)\\
&= \sup_{j\geq i} \pi_i(\pi_j)^{\wedge} \circ \big(\pi_{j+1}(a)\big) \circ \pi_j(\pi_j)^{\wedge} \circ \Big(\pi_{j+1}\big((\pi_i)^{\wedge}(y)\big)\Big) \circ \pi_j(\pi_i)^{\wedge}(y)\\
&= \sup_{j\geq i} p_i^j \circ \big(\pi_{j+1}(a)\big) \circ \Big(\big(p_i^{j+1}\big)^{\wedge}(y)\Big) \circ \big(p_i^j\big)^{\wedge}(y)\\
&= \sup_{j\geq i} p_i^j \circ \big(\pi_{j+1}(a)\big) \circ \big(p_{i-1}^j\big)^{\wedge}\, y p_{i-1}^j \big(p_i^j\big)^{\wedge}(y)\\
&= \sup_{j\geq i} p_i^j \circ \big(\pi_{j+1}(a)\big) \circ \big(p_i^j\big)^{\wedge}\big(p_{i-1}^i\big)^{\wedge}\, y p_{i-1}^i p_i^j \big(p_i^j\big)^{\wedge}(y)\\
&= \sup_{j\geq i}\Big(p_{i+1}^{j+1}\big(\pi_{j+1}(a)\big)\Big) \circ \big(p_{i-1}^i\big)^{\wedge}\, y p_{i-1}^i(y)\\
&= \sup_{j\geq i}\big(\pi_{i+1}(a)\big) \circ \Big(\big(p_i^{i+1}\big)^{\wedge}(y)\Big)(y)\\
&= g_{i+1,a}(y)\\
&= \pi_{i+1}\big(g(a)\big)(y).
\end{aligned}$$

Thus $\pi_i f(\pi_i)^{\wedge} = \pi_{i+1}(g)$, and so

$$\begin{aligned}
\Phi^{-1}(f) &= \sup_{i\geq 0}(\pi_{i+1})^{\wedge}\,\pi_{i+1}(g(a))\\
&= g(a) \qquad (\text{ by Lemma 2.7.3 }).
\end{aligned}$$

Therefore $[\![\lambda x.y(xx)]\!]\rho\left\{\frac{a}{y}\right\} = g(a)$.

It follows that $h(a) = \Phi(g(a))(g(a))$ for $a \in P_\infty$. Thus for $y \in P_i$,

$$\begin{aligned}\pi_i h(\pi_i)^\frown(y) &= \pi_i\Big(\Phi\big(g((\pi_i)^\frown(y))\big)\big(g((\pi_i)^\frown(y))\big)\Big)\\ &= \pi_i\Big(\sup_{j\geq 0}(\pi_j)^\frown \circ \Big(\pi_{j+1}\big(g((\pi_i)^\frown(y))\big)\Big) \circ \pi_j\big(g((\pi_i)^\frown(y))\big)\Big)\\ &= \sup_{j\geq 0} \pi_i(\pi_j)^\frown \circ \Big(\pi_{j+1}\big(g((\pi_i)^\frown(y))\big)\Big)\Big(\pi_j\big(g((\pi_i)^\frown(y))\big)\Big)\\ &= \sup_{j\geq 0} \pi_i(\pi_j)^\frown \circ \beta_{j+1,i}(\beta_{j,i}),\end{aligned}$$

where

$$\beta_{k,i} = \pi_k\big(g((\pi_i)^\frown(y))\big)$$

for $k \geq 0$.

Let $i \geq 2$, and $y \in P_i$. Let $q_k = (p_k^{i-1})^\frown p_k^{i-1} y$ for $0 \leq k < i$. Let $q_l^m = q_m q_{m-1} \cdots q_l$ for $0 \leq l \leq m < i$. Let

$$\alpha_{i,j} = \pi_i(\pi_j)^\frown \circ \beta_{j+1,i}(\beta_{j,i}).$$

Next we calculate $\alpha_{i,j}$ for $j \geq 0$. Now

$$\begin{aligned}\alpha_{i,0} &= \pi_i(\pi_0)^\frown \circ \beta_{1,i}(\beta_{0,i})\\ &= (p_0^i)^\frown \circ g_{1,(\pi_i)^\frown(y)}(\beta_{0,i})\\ &= (p_0^i)^\frown \circ g_{i,(\pi_i)^\frown(y)}(p_0^i(y))\\ &= (p_0^i)^\frown \circ \big(\pi_1((\pi_i)^\frown(y))\big) \circ \big((p_0^1)^\frown(p_0^i(y))\big)(p_0^i(y))\\ &= (p_0^i)^\frown \circ (p_1^i(y))(p_0^i(y))\\ &= (p_0^i)^\frown p_0^{i-1} y (p_0^{i-1})^\frown p_0^i(y)\\ &= (p_{i-1}^i)^\frown (p_0^{i-1})^\frown p_0^{i-1} y (p_0^{i-1})^\frown p_0^1 p_1^i(y)\\ &= (p_{i-1}^i)^\frown q_0 (p_0^{i-1})^\frown (p_1^i(y)(0))\\ &= (p_{i-1}^i)^\frown q_0 (p_0^{i-1})^\frown p_0^{i-1} y (p_0^{i-1})^\frown (0)\\ &= (p_{i-1}^i)^\frown q_0^2(0_{i-1}),\end{aligned}$$

and

$$
\begin{aligned}
\alpha_{i,1} &= \pi_i(\pi_i)^{\frown} \circ \beta_{2,i}(\beta_{1,i}) \\
&= (p_1^i)^{\frown} \circ g_{2,(\pi_i)^{\frown}(y)}(\beta_{1,i}) \\
&= (p_1^i)^{\frown} \circ \left(\pi_2((\pi_i)^{\frown}(y))\right) \circ \left((p_1^2)^{\frown}(\beta_{j,i})i\right)(\beta_{j,i}) \\
&= (p_1^i)^{\frown} \circ (p_2^i(y)) \circ (p_0^1)^{\frown} \circ (\pi_1(\beta_{1,i})) \circ p_0^1(\beta_{1,i}) \\
&= (p_1^i)^{\frown} p_1^{i-1} y (p_1^{i-1})^{\frown} (p_0^1)^{\frown} \circ \beta_{1,i}(\beta_{0,i}) \\
&= (p_{i-1}^i)^{\frown} (p_1^{i-1})^{\frown} p_1^{i-1} y (p_0^{i-1})^{\frown} \circ \beta_{1,i}(\beta_{0,i}) \\
&= (p_{i-1}^i)^{\frown} q_1 p_{i-1}^i (p_0^i)^{\frown} \circ \beta_{1,i}(\beta_{0,i}) \\
&= (p_{i-1}^i)^{\frown} q_1 p_{i-1}^i (\alpha_{i,0}) \\
&= (p_{i-1}^i)^{\frown} q_1 p_{i-1}^i (p_{i-1}^i)^{\frown} q_0^2 (0_{i-1}) \\
&= (p_{i-1})^{\frown} q_1 q_0^2 (0_{i-1}) \\
&= (p_{i-1}^i)^{\frown} q_0^1 q_0 (0_{i-1}).
\end{aligned}
$$

We now show that for $1 \leq i \leq i-1$, $\alpha_{i,j} = (p_{i-1}^i)^{\frown} q_0^j q_0(0_{i-1})$. We proceed by induction.

Assume that $\alpha_{i,j-1} = (p^i_{i-1})^\wedge q_0^{j-1} q_0(0_{i-1})$. Then

$$\begin{aligned}
\alpha_{i,j} &= \pi_i(\pi_j)^\wedge g_{j+1,(\pi_i)^\wedge(y)}(\beta_{j,i}) \\
&= (p^i_j)^\wedge \circ \left(\pi_{j+1}((\pi_i)^\wedge(y))\right) \circ \left((p^{j+1}_j)^\wedge(\beta_{j,i})\right)(\beta_{j,i}) \\
&= (p^i_j)^\wedge \circ (p^i_{j+1}(y)) \circ (p^j_{j-1})^\wedge \circ \beta_{j,i} \circ p^j_{j-1}(\beta_{j,i}) \\
&= (p^i_j)^\wedge p_j^{j-1} y (p^{i-1}_j)^\wedge (p^j_{j-1})^\wedge \circ \beta_{j,i}(\pi_{j-1}(\beta_{j-1,i})) \\
&= (p^i_{i-1})^\wedge (p^{i-1}_j)^\wedge p^{i-1}_j y p^i_{i-1} (p^i_{i-1})^\wedge (p^{i-1}_{j-1})^\wedge \circ \beta_{j,i}(\beta_{j-1,i}) \\
&= (p^i_{i-1})^\wedge q_j p^i_{i-1} (p^i_{j-1})^\wedge \circ \beta_{j,i}(\beta_{j-1,i}) \\
&= (p^i_{i-1})^\wedge q_j p^i_{i-1} \alpha_{i,j-1} \\
&= (p^i_{i-1})^\wedge q_j p^i_{i-1} (p^i_{i-1})^\wedge q_0^{j-1} q_0(0_{i-1}) \\
&= (p^i_{i-1})^\wedge q_j q_0^{j-1} q_0(0_{i-1}) \\
&= (p^i_{i-1})^\wedge q_0^j q_0(0_{i-1}).
\end{aligned}$$

Now

$$\begin{aligned}
\alpha_{i,i} &= \pi_i(\pi_i)^\wedge \circ \beta_{i+1,i}(\beta_{i,i}) \\
&= g_{i+1,(\pi_i)^\wedge(y)}(\beta_{i,i}) \\
&= \left(\pi_{i+1}((\pi_i)^\wedge(y))\right) \circ \left((p^{i+1}_i)^\wedge(\beta_{i,i})\right)(\beta_{i,i}) \\
&= ((p^{i+1}_i)^\wedge(y)) \circ (p^i_{i-1})^\wedge \circ \beta_{i,i} \circ p^i_{i-1}(\beta_{i,i}) \\
&= (p^i_{i-1})^\wedge y p^i_{i-1} (p^i_{i-1})^\wedge \circ \beta_{j,i}(\beta_{i-1,i}) \\
&= (p^i_{i-1})^\wedge y p^i_{i-1} \alpha_{i,i-1} \\
&= (p^i_{i-1})^\wedge y p^i_{i-1} (p^i_{i-1})^\wedge q_0^{i-1} q_0(0_{i-1}) \\
&= (p^i_{i-1})^\wedge y q_0^{i-1} q_0(0_{i-1}) \\
&= (p^i_{i-1})^\wedge y q_{i-1} q_0^{i-2} q_0(0_{i-1}) \\
&= (p^i_{i-1})^\wedge y (p^{i-1}_{i-1})^\wedge p^{i-1}_{i-1} y q_0^{i-2} q_0(0_{i-1}) \\
&= (p^i_{i-1})^\wedge y^2 q_0^{i-2} q_0(0_{i-1}).
\end{aligned}$$

Finally, we show that for $j \geq i$, $\alpha_{i,j} = \left(p^i_{i-1}\right)^\wedge y^{j-i+2} q_0^{i-2} q_0(0_{i-1})$. We proceed by induction. Assume that $\alpha_{i,j-1} = \left(p^i_{i-1}\right)^\wedge y^{j-i+1} q_0^{i-2} q_0(0_{i-1})$ for $j > i$. Then

$$\begin{aligned}
\alpha_{i,j} &= \pi_i(\pi_j)^\wedge \circ \beta_{j+1,i}(\beta_{j,i}) \\
&= p^j_i g_{j+1,(\pi_i)^\wedge(y)}(\beta_{j,i}) \\
&= p^j_i \circ \left(\pi_{j+1}((\pi_i)^\wedge(y))\right) \circ \left((p^{j+1}_j)^\wedge(\beta_{j,i})\right)(\beta_{j,i}) \\
&= p^j_i \circ \left((p^{j+1}_i)^\wedge(y)\right) \circ (p^j_{j-1})^\wedge \circ \beta_{j,i} \circ p^j_{j-1}(\beta_{j,i}) \\
&= p^j_i (p^j_{i-1})^\wedge y p^j_{i-1} (p^j_{j-1})^\wedge \circ \beta_{j,i}\left(\pi_{j-1}(\beta_{j,i})\right) \\
&= (p^i_{i-1})^\wedge y p^{j-1}_{i-1} \circ \beta_{j,i}(\beta_{j-1,i}) \\
&= (p^i_{i-1})^\wedge y p^j_{i-1} p^{j-1}_i \circ \beta_{j,i}(\beta_{j-1,i}) \\
&= (p^i_{i-1})^\wedge y p^i_{i-1}(\alpha_{i,j-1}) \\
&= (p^i_{i-1})^\wedge y p^i_{i-1} y^{j-i+1} q_0^{i-2} q_0(0_{i-2}) \\
&= (p^i_{i-1})^\wedge y^{j-i+2} q_0^{i-2} q_0(0_{i-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\pi_i h(\pi_i)^\wedge(y) &= \sup_{j \geq 0} \alpha_{i,j} \\
&= \sup_{j \geq 0} (p^i_{i-1})^\wedge y^{j-i+2} q_0^{i-2} q_0(0_{i-1}) \\
&= (p^i_{i-1})^\wedge \left(\sup_{j \geq i} y^{j-i+2} q_0^{i-2}(0_{i-1})\right).
\end{aligned}$$

Now

$$\begin{aligned}
q_k &= (p^{i-1}_k)^\wedge p^{i-1}_k y \\
&= (p^{k+1}_k p^{i-1}_{k+1})^\wedge p^{k+1}_k pk + 1^{i-1} y \\
&= (p^{i-1}_{k+1})^\wedge (p^{k+1}_k)^\wedge p^{k+1}_k p^{i-1}_{k+1} y \\
&\leq (p^{i-1}_{k+1})^\wedge p^{i-1}_{k+1} y \\
&= q_{k+1}
\end{aligned}$$

for $0 \leq k < i - 1$. Furthermore, $q_{i-2} = (p^{i-1}_{i-2})^\wedge p^{i-1}_{i-2} y \leq y$. Therefore, by Proposition 3.4.2, $\sup_{j \geq i} y^{j-i+2} q_0^{i-2} q_0(0_{i-1})$ is the least fixed point of y, and is therefore equal to

$\sup_{j\geq 0} y^j(0_{i-1})$. Thus $\pi_i h(\pi_i)\hat{}\,(y) = F_{i+1}(y)$ for all $y \in P_i$, and so $\pi_i h(\pi_i)\hat{} = F_{i+1}$ for $i \geq 2$. Therefore

$$[\![\mathbf{Y}]\!]\rho = \sup_{i\geq 0} (\pi_{i+1})\hat{}\,\pi_{i+1}(F) = F.$$

According to Proposition 1.2.5, $\Phi([\![\mathbf{Y}]\!])(a)$ is a fixed point of $\Phi(a)$ for all $a \in P_\infty$. We now show that $\Phi([\![\mathbf{Y}]\!])$ is the least fixed of $\Phi(a)$. Let y be a fixed point of $\Phi(a)$. Then $y = (\Phi(a))^j(y)$ for all $j \geq 0$. Thus

$$\begin{aligned}
y &= \sup_{j\geq 0}(\Phi(a))^j(y) \\
&= \sup_{j\geq 0}\Big(\big(\sup_{i\geq 0}(\pi_i)\hat{} \circ (\pi_{i+1}(a)) \circ \pi_i\big)^j(y)\Big) \qquad (\text{ by Lemma 2.8.1 }) \\
&= \sup_{j\geq 0}\Big(\sup_{i\geq 0}\big((\pi_i)\hat{} \circ (\pi_{i+1}(a)) \circ \pi_i\big)(y)\Big) \\
&= \sup_{j\geq 0}\Big(\sup_{i\geq 0}(\pi_i)\hat{} \circ (\pi_{i+1}(a))^j \circ \pi_i\Big)(y) \\
&= \sup_{j\geq 0}\Big(\sup_{i\geq 0}(\pi_i)\hat{} \circ (\pi_{i+1}(a))^j \circ \pi_i(y)\Big) \\
&= \sup_{i\geq 0}\Big(\sup_{j\geq 0}(\pi_i)\hat{} \circ (\pi_{i+1}(a))^j \circ \pi_i(y)\Big) \qquad (\text{ by Lemma 2.8.1 }) \\
&= \sup_{i\geq 0}\Big((\pi_i)\hat{} \circ \big(\sup_{j\geq 0}(\pi_{i+1}(a))^j \circ \pi_i(y)\big)\Big) \\
&\geq \sup_{i\geq 0}\Big(\sup_{j\geq 0}(\pi_{i+1}(a))^j(0_i)\Big) \\
&= \sup_{i\geq 0}(p_i^{i+1})\hat{}\Big(\sup_{j\geq 0}(\pi_{i+1}(a))^j(0_i)\Big) \\
&= \sup_{i\geq 0}(\pi_{i+1})\hat{}\Big(F_{i+2}(\pi_{i+1}(a))\Big) \\
&= \sup_{i\geq 0}(\pi_{i+1})\hat{}\,F_{i+2}\pi_{i+1}(a) \\
&= \Phi(F)(a) \\
&= \Phi([\![\mathbf{Y}]\!])(a).
\end{aligned}$$

Thus $\Phi([\![\mathbf{Y}]\!])(a)$ is the least fixed point of $\Phi(a)$.

Chapter 4

Combinatorial Semigroups

We investigate two submonoids of the monoid of combinators under composition in the extensional λ-calculus. In particular, we show that this monoid contains the non-negative integers under addition, and a free monoid on a infinite number of generators as submonoids.

§ 1. Introduction

The set Λ_0 of combinators in the extensional λ-calculus forms a monoid under the operation of "composition". Let A and B be combinators. Then we define $A \circ B = \lambda x.A(Bx)$. Even in the non-extensional case, this operation makes Λ_0 into a semigroup. We check associativity as follows:

$$\begin{aligned}(A \circ B) \circ C &= \lambda x.(A \circ B)(Cx) \\ &= \lambda x.(\lambda y.A(By))(Cx) \\ &= \lambda x.(A(By))[y \leftarrow Cx] \\ &= \lambda x.A(B(Cx)),\end{aligned}$$

and

$$\begin{aligned}A \circ (B \circ C) &= \lambda x.A((B \circ C)x) \\ &= \lambda x.A((\lambda y.B(Cy))x) \\ &= \lambda x.(A(B(Cy)))[y \leftarrow x] \\ &= \lambda x.A(B(Cx)).\end{aligned}$$

In the extensional λ-calculus we make the following calculations:

$$\begin{aligned}\mathbf{I} \circ A &= \lambda x.\mathbf{I}(Ax) \\ &= \lambda x.Ax \\ &= A, \qquad \text{(by } \eta\text{-reduction)}\end{aligned}$$

and

$$\begin{aligned}A \circ \mathbf{I} &= \lambda x.A(\mathbf{I}x) \\ &= \lambda x.Ax \\ &= A.\end{aligned}$$

Thus Λ_0 is a monoid under $\circ$ in the extensional λ-calculus. From this point on, we assume extensionality.

§ 2. First Projections

Let P be the set of combinators of the form $\pi_1^n = \lambda x_1 \ldots x_n.x_1$. We may think of these combinators as defining "projections", since $\pi_1^n A_1 \ldots A_n = A_1$. We compute the operation of composition on this set:

$$\begin{aligned}\pi_1^m \circ \pi_1^n &= (\lambda x_1 \ldots x_m.x_1) \circ (\lambda y_1 \ldots y_n.y_1)\\ &= \lambda z.(\lambda x_1 \ldots x_m.x_1)((\lambda y_1 \ldots y_n.y_1)z)\\ &= \lambda z.(\lambda x_1 \ldots x_m.x_1)(\lambda y_2 \ldots y_m.z)\\ &= \lambda z.(\lambda x_2 \ldots x_m.(\lambda y_2 \ldots y_n.z))\\ &= \lambda x_1 \ldots x_{m+n-1}.x_1\\ &= \pi_1^{m+n-1}.\end{aligned}$$

Thus the function $\pi_1^n \mapsto n-1$ is a monoid isomorphism onto the non-negative integers under addition.

§ 8. Binary Trees

Let X be the set of combinators defined recursively by:

(1) $\lambda x.x \in X$

(2) $\lambda x.A \in X$ and $\lambda x.B \in X$ imply $\lambda x.AB \in X$.

Thus X is the set of combinators containing only the variable x and a single λ. In X the operation $\circ$ has a simple form:

$$\begin{aligned}(\lambda x.A) \circ (\lambda x.B) &= \lambda x.(\lambda y.A)((\lambda x.B)x)\\ &= \lambda x.(\lambda y.A)B\\ &= \lambda x.A[x \leftarrow B].\end{aligned}$$

We show that $(X, \circ)$ is a free monoid on infinitely many generators. We do this by showing that $(X, \circ)$ is isomorphic to a monoid of binary trees.

The free monoid on the set $\{0,1\}$ is denoted by $\{0,1\}^*$. The empty string is denoted by ϵ. For $x \in \{0,1\}$, $|x|$ denotes the length of the string x. We recall that free monoids satisfy the property of equidivisibility: if $ab = cd$ then either there exists u such that $au = c$ and $ud = b$, or there exists v such that $cv = a$ and $vb = d$. If $xy = z$ then x is a left prefix for z. If $A, B \subseteq \{0,1\}^*$ then $AB = \{ab : a \in A \text{ and } b \in B\}$.

A **binary tree** (or **tree**) is a finite subset of $\{0,1\}^*$ satisfying the following properties:

(T1) There exists $r \in T$ such that for all $x \in T$, $x = ry$ for some $y \in \{0,1\}^*$.

(T2) $rxy \in T$ implies $rx \in T$.

(T3) $rx0 \in T$ if and only if $rx1 \in T$.

Note that there is at most one r satisfying (T1). The r satisfying (T1) is called the **root** of T and is denoted $r(T)$. A **leaf** of a binary tree T is an element x of T such that $xy \in T$ implies $y = \epsilon$. $L(T)$ denotes the set of leaves of T. $\mathcal{B}$ denotes the set of trees with root ϵ. Note that a finite subset T of $\{0,1\}^*$ is a tree with root ϵ if and only if it satisfies:

(T4) $\epsilon \in T$.

(T5) $xy \in T$ implies $x \in T$.

(T6) $x0 \in T$ if and only if $x1 \in T$.

One should think of a binary tree in the usual way, where the partial order is given by $x \leq y$ when y is a left prefix for x. A final 0 indicates a branch to the left, and a 1 indicates a branch to the right. For example, the tree $\epsilon, 0, 1, 00, 01$ is drawn below.

```
            ε
          /   \
        0       1
      /   \
    00     01
```

Let $S, T \in \mathcal{B}$. Then we define $S * T = S \bigcup L(S)T$. We show that $\mathcal{B}$ is a monoid under the operation $*$. Graphically, this operation consists of grafting B onto every leaf of A.

Proposition 8.1: If $S, T \in \mathcal{B}$ then $S * T \in \mathcal{B}$.

Proof: Since $\epsilon \in S * T$, (T4) is satisfied. We show that (T5) is satisfied. Let $xy \in S * T$.

Case 1: Let $xy \in S$. Then $x \in S$.

Case 2: Let $xy = st$, where $s \in L(S)$, $t \in T$, and $t \neq \epsilon$.

Case 2a: Assume that there exists u such that $xu = s$. Then $xu \in S$, and so $x \in S$.

Case 2b: Assume that there exists v such that $sv = x$ and $vy = t$. Thus $vy \in T$ and so $v \in T$. Hence $sv \in L(S)T$.

In any case, $x \in S * T$. Last, we must show that (T6) is satisfied. Let $x0 \in S * T$.

Case 1: Let $x0 \in S$. Then $x1 \in S$.

Case 2: Let $x0 = st$, where $s \in L(S)$, $t \in T$, and $t \neq \epsilon$.

Case 2a: Let $t = 0$. Then $x = s$. Also $0 \in T$ implies $1 \in T$ and so $x1 \in L(S)T$.

Case 2b: Let $t = u0$, where $u \neq \epsilon$. Then $x = su$. Also $u1 \in T$, and hence $x1 = su1 \in L(S)T$.

Thus $x0 \in S * T$ implies $x1 \in S * T$. The converse is similar. ♣

Lemma 8.2: Let $x \in T$, where $T \in \mathcal{B}$. Then there exists $y \in L(T)$ such that x is a left prefix for y.

Proof: Choose $y \in T$ such that x is a left prefix for y and y is of maximal length with respect to this property. Then if $yu \in T$, $|y| \leq |yu|$ and x a left prefix for yu imply that $|yu| = |y|$. Hence $u = \epsilon$. ♣

Lemma 8.3: Let $T \in \mathcal{B}$ and $x, y \in L(T)$. If there exist u and v such that $xu = yv$, then $x = y$.

Proof: If there exist u and v such that $xu = yv$, then by equidivisibility either there exists r such that $xr = y$ or there exists s such that $ys = x$. In the first case we must have $r = \epsilon$, and in the second $s = \epsilon$. Thus in either case, $x = y$. ♣

Lemma 8.4: Let $S, T \in \mathcal{B}$. Then $L(S * T) = L(S)L(T)$.

Proof: Let $s \in L(S)$ and $t \in L(T)$. Then $st \in S * T$. We claim that $st \in L(S * T)$. Let $stu \in S * T$.

Case 1: Let $stu \in S$. Then $tu = \epsilon$ and so $u = \epsilon$.

Case 2: Let $stu = xy$, where $x \in L(S)$ and $y \in T$. Then by Lemma 3, $s = x$. Thus $tu = y$, and so $u = \epsilon$.

Thus $L(S)L(T) \subseteq L(S * T)$.

Now let $x \in L(S * T)$. We must show that $x \in L(S)L(T)$.

Case 1: Let $x \in S$. Then we claim that $\epsilon \in L(T)$. (I.e., that $T = \{\epsilon\}$.) Let $t \in T$. Choose $s \in L(S)$ such that x is a left prefix for s. Then $st \in S * T$. Let u be such that $xu = s$. Then $xut \in S * T$, and hence $ut \in \epsilon$. Thus $t = \epsilon$. Hence $T = \{\epsilon\}$ and so $\epsilon \in L(T)$. Furthermore $u = \epsilon$, and so $x \in L(S)$. Thus $x = x\epsilon \in L(S)L(T)$.

Case 2: Let $x = st$, where $s \in L(S)$ and $t \in T$. If $tu \in T$ then $xu = stu \in S * T$, and hence $u = \epsilon$. Thus $t \in L(T)$ and so $x \in L(S)L(T)$.

Thus $L(S * T) \subseteq L(S)L(T)$. ♣

Theorem 3.5: $(\mathcal{B}, *)$ is a monoid.

Proof: Clearly $\{\epsilon\}$ is an identity, so we only need to check associativity.

$$\begin{aligned} S*(T*U) &= S \bigcup L(S)(T*U) \\ &= S \bigcup L(S)(T \bigcup L(T)U) \\ &= S \bigcup L(S)T \bigcup L(S)L(T)U \\ &= S*T \bigcup L(S*T)U \\ &= (S*T)*U. \clubsuit \end{aligned}$$

Theorem 3.6: $(\mathcal{B}, *)$ is a cancellative monoid.

Proof: Let $A, B, C \in \mathcal{B}$, and $A*B = A*C$. We must show that $B = C$. Let $b \in B$. Then for any $a \in L(A)$, we have $ab \in A*B$ and hence $ab \in A*C$.

Case 1: Let $ab \in A$. Then $b = \epsilon \in C$.

Case 2: Assume that there exist $a' \in L(A)$ and $c \in C$ such that $ab = a'c$. But then $a = a'$ by Lemma 3. Thus $b = c \in C$.

Hence $B \subseteq C$. Similarly, $C \subseteq B$.

Now let $A*B = C*B$. We must show that $A = C$. Let $a \in L(A)$ and choose $b \in B$ of maximal length. Then $b \in L(B)$. We have $ab \in L(A)L(B) = L(A*B) = L(C*B) = L(C)L(B)$. Hence $ab = cb'$, where $c \in L(C)$ and $b' \in L(B)$.

Case 1: Assume that there exists u such that $a = cu$ and $ub = b'$. Then since b is of maximal length, $b = b'$. Thus $u = \epsilon$ and hence $a = c \in C$.

Case 2: Assume that there exists v such that $c = av$ and $vb' = b$. Then $a \in C$.

In either case, $a \in C$. Hence $L(A) \subseteq C$ and so $A \subseteq C$. Similarly, $C \subseteq A$. $\clubsuit$

Proposition 3.7: $\mathcal{B}$ is an equidivisible monoid.

Proof: Let $A, B, C, D \in \mathcal{B}$ and $A*B = C*D$. We first show that either $A \subseteq C$ or $C \subseteq A$. We proceed by contradiction. Suppose that $a \in L(A) \setminus C$ and $c \in L(C) \setminus A$. Choose b of maximal length in B. Then $b \in L(B)$. Thus $ab \in L(A)L(B) = L(C)L(D)$, so there exist $c' \in L(C)$ and $d \in L(D)$ such that $ab = c'd$. Since a is not an element of C, there exists $v \neq \epsilon$ such that $c'v = a$ and $d = vb$. Then $|d| > |b|$. Now $cd = a'b'$, where $a' \in L(A)$ and $b' \in L(B)$. Since c is not in A, there exists $u \neq \epsilon$ such that $a'u = c$ and $ud = b'$. Thus $|b'| > |d| > |b|$. But this contradicts the choice of b. Hence either $A \subseteq C$ or $C \subseteq A$.

Thus we may suppose without loss of generality that $A \subseteq C$. Let $S = \{x : yx \in C \text{ for all } y \in L(A)\}$. We claim that $S \in \mathcal{B}$. Let $y \in L(A)$. Then $\epsilon y \in A$ and hence $\epsilon y \in C$. Thus $\epsilon \in S$. Hence (T4) is satisfied. Let $xy \in S$. Then $uxy \in C$ for all $u \in L(A)$. Hence $ux \in C$ for all $u \in L(A)$ and hence $x \in S$. Thus (T5) is satisfied. If $x0 \in S$ then $yx0 \in C$ for all $y \in L(A)$. Thus $yx1 \in C$ for all $y \in L(A)$, and so $x1 \in S$. Hence $x0 \in S$ implies $x1 \in S$. The converse is similar. Hence (T6) is satisfied and so $S \in \mathcal{B}$.

We now claim that $A * S = C$. Let $x \in A * S$.

Case 1: Let $x \in A$. Then $x \in C$.

Case 2: Let $x = as$, where $a \in L(A)$ and $s \in S$. Then $as \in C$.

Hence $A * S \subseteq C$.

Let $c \in L(C)$. Choose d of maximal length in D. Then $d \in L(D)$. Thus $cd = ab$, where $a \in L(A)$ and $b \in L(B)$.

Case 1: Assume that there exists w such that $cw = a$ and $wb = d$. Then $c \in A$, and hence $c \in A * S$.

Case 2: Assume that there exists u such that $au = c$ and $ud = b$. We show that $u \in S$. Let $a' \in L(A)$. Then $a'b = c'd'$, where $c' \in L(C)$ and $d' \in L(D)$. Thus $a'ud = c'd'$.

Case 2a: Assume that there exists z such that $zd = d'$ and $c'z = a'u$. Since d is of maximal length, we have $z = \epsilon$. Hence $a'u = c' \in C$.

Case 2b: Assume that there exists v such that $vd' = d$ and $a'uv = c'$. Then $a'uv \in C$ and hence so $a'u \in C$.

In either case $a'u \in C$, and hence $u \in S$. Thus $c = au \in A * S$. So $L(C) \subseteq A * S$ and hence $C \subseteq A * S$.

Thus $C = A * S$. Now $A * B = A * S * D$, and so $B = S * D$. ♣

Lemma 8.8: A monoid M is free if and only if M is cancellative, M is equidivisible, M has a trivial group of units, and every element of M has at most a finite number of nontrivial left factors.

Proof: See Lallement [1979]. ♣

Theorem 8.9: $\mathcal{B}$ is a free monoid on an infinite number of generators.

Proof: $\mathcal{B}$ is cancellative and equidivisible. It is clear that $\mathcal{B}$ has a trivial group of units, and that each element of $\mathcal{B}$ has at most a finite number of nontrivial left factors. Hence $\mathcal{B}$ is free. Let $\mathcal{B}^+ = \mathcal{B} \setminus \{ \{\epsilon\} \}$. Then $\mathcal{B}^+ \setminus (\mathcal{B}^+)^2$ is the unique minimal generating set for $\mathcal{B}$ (Lallement [1979]). Consider $T_A = \{\epsilon, 1\} \bigcup 0A$, where $A \in \mathcal{B}$. Then $T_A \in \mathcal{B}$, and we claim that T_A does not factor. $1 \in L(T_A)$. Hence if $T_A = C * D$, then $1 \in L(C) * L(D)$. Hence either $\epsilon \in L(C)$ or $\epsilon \in L(D)$. Thus either $C = \{\epsilon\}$ or $D = \{\epsilon\}$. Thus $\{ T_A : A \in \mathcal{B} \} \subseteq \mathcal{B}^+ \setminus (\mathcal{B}^+)^2$, and clearly this is an infinite set. ♣

Lemma 3.10: Let T be a tree. We define $S(T, x)$ for $x \in T$ as follows: $S(T, x) = \{ y : xy \in T \}$. Then $S(T, x) \in \mathcal{B}$.

Proof: Clearly $x\epsilon \in T$, so $\epsilon \in S(T, x)$. Thus (T4) is satisfied. Let $uv \in S(T, x)$. Then $xuv \in T$, and so $xu \in T$. Hence $u \in S(T, x)$. Thus (T5) is satisfied. Let $u0 \in S(T, x)$. Then $xu0 \in T$. Thus $xu1 \in T$, and hence $u1 \in S(t, x)$. Thus $u0 \in S(T, x)$ implies $u1 \in S(T, x)$. The converse is similar. Thus (T6) is satisfied, and so $S(T, x) \in \mathcal{B}$. ♣

Strictly speaking, we would not want to call $S(T, x)$ a subtree of T since in general we do not have $S(T, x) \subseteq T$. However, it does make sense to call $xS(T, x)$ the subtree of T originating at the node x.

Lemma 3.11: Let $A, B \in \mathcal{B}$. Then $T(r, A, B) = \{ r \} \bigcup r0A \bigcup r1B$ is a tree with root r.

Proof: Let $T = T(r, A, B)$. Clearly r is a left prefix for every element of T. Thus (T1) is satisfied. Let $rxu \in T$.

Case 1: Let $x = \epsilon$. Then $rx = r \in T$.

Case 2: Let $x = 0y$. Then $r0yu \in T$. Hence $r0yu \in r0A$, and so $yu \in A$. Thus $y \in A$, and so $rx = r0y \in r0A$.

Case 3: Let $x = 1y$. This case is similar to case 2.

Thus (T2) is satisfied. Let $rx0 \in T$.

Case 1: Let $x = \epsilon$. Then $rx1 = r1 \in T$.

Case 2: Let $x = 0y$. Then $r0y0 \in T$, and so $r0y0 \in r0A$. Thus $y0 \in A$, and so $y1 \in A$. Thus $rx1 = r0y0 \in r0A$.

Case 3: Let $x = 1y$. This case is similar to case 2.

Thus $rx0 \in T$ implies $rx1 \in T$. The converse is similar. Thus (T3) is satisfied. ♣

Lemma 3.12: : Let $A, B \in \mathcal{B}$. Then $L(T(r, A, B)) = r0L(A) \bigcup r1L(B)$.

Proof: Let $y \in L(T(r, A, B))$. Then $y \in T(r, A, B)$.

Case 1: Let $y = r$. Since $\epsilon \in A$, $r0\epsilon \in T(r, A, B)$. Thus $0\epsilon = \epsilon$. Hence this case cannot occur.

Case 2: Let $y = r0A$, where $a \in A$. We claim that $a \in L(A)$. Suppose $au \in A$. Then $r0au \in r0A$ and so $r0au \in T(r, A, B)$. Thus $u = \epsilon$. Hence $a \in L(A)$, and so $y \in r0L(A)$.

Case 3: Let $y = r1b$, where $b \in B$. This case is similar to case 2.

Thus $L(T(r, A, B)) \subseteq r0L(A) \bigcup r1L(B)$. Let $a \in L(A)$. Then $r0A \in t(r, A, B)$. Suppose $r0au \in T(r, A, B)$. Clearly $r0au \neq r$, and is not in $r1B$. Thus $r0au \in r0A$, and so $au \in A$. Hence $u = \epsilon$. Thus $r0a \in L(T(r, A, B))$. Hence $r0L(A) \subseteq L(T(r, A, B))$. Similarly $r1L(B) \subseteq L(T(r, A, B))$. ♣

Lemma 3.13: Let T be a tree with root r. Then xT is a tree with root xr, and $L(xT) = xL(T)$.

Proof: Let $y \in xT$. Then $y = xz$, with $z \in T$, and so $z = ru$. Thus $y = xru$. Thus xr is a left prefix for y. Thus (T1) is satisfied. Let $xryu \in xT$. Then $ryu \in T$, so $ry \in T$, and so $xry \in xT$. Thus (T2) is satisfied. Let $xry0 \in xT$. Then $ry0 \in T$, and so $ry1 \in T$. Hence $xry1 \in xT$. Thus $xry0 \in xT$ implies $xry1 \in xT$. The converse is similar. Hence (T3) is satisfied. Thus xT is a tree with root xr.

Let $y \in L(xT)$. Then $y = xt$, with $t \in T$. If $tu \in T$, then $xty \in xT$. Thus $yu \in xT$, so $u = \epsilon$. Hence $t \in L(T)$, and so $y \in xL(T)$. Thus $L(xT) \subseteq xL(T)$. Let $t \in L(T)$. Then if $xtu \in xT$, $tu \in T$. Hence $u = \epsilon$. Thus $xt \in L(xT)$. Hence $xL(T) \subseteq L(xT)$. ♣

Lemma 3.14: Let $A, B, C \in \mathcal{B}$. Then $T(\epsilon, A, B) * C = T(\epsilon, A * C, B * C)$.

Proof: We calculate as follows:

$$\begin{aligned} T(\epsilon, A, B) * C &= (\{\epsilon\} \bigcup 0A \bigcup 1B) * C \\ &= \{\epsilon\} \bigcup 0A \bigcup 1B \bigcup L(\{\epsilon\} \bigcup 0A \bigcup 1B)C \\ &= \{\epsilon\} \bigcup 0A \bigcup 1B \bigcup 0L(A)C \bigcup 1L(B)C \\ &= \{\epsilon\} \bigcup 0(A \bigcup L(A)C) \bigcup 1(B \bigcup L(B)C) \\ &= \{\epsilon\} \bigcup 0A * C \bigcup 1B * C \\ &= T(\epsilon, A * C, B * C). \ ♣ \end{aligned}$$

Lemma 3.15: Let $T \in \mathcal{B}$, with $T \neq \{\epsilon\}$. Then T has a unique representation of the form $T = T(\epsilon, A, B)$, where $A, B \in \mathcal{B}$.

Proof: Let $A = S(T,0)$ and $B = S(T,1)$. Then A and B are in $\mathcal{B}$. Clearly $T(\epsilon, A, B) \subseteq T$. Let $x \in T$.

Case 1: Let $x = \epsilon$. Then $x \in \{\epsilon\}$.

Case 2: Let $x = 0y$. Then $y \in S(T,0)$, so $x \in 0A$.

Case 3: Let $x = 1y$. Then $y \in S(T,1)$, so $x \in 1A$. In any case $x \in T(\epsilon, A, B)$. Thus $T \subseteq S(\epsilon, A, B)$. Suppose that also $T = S(\epsilon, C, D)$. Let $a \in A$. Then $0a \in 0A$, so $0a \in T(\epsilon, C, D)$. Hence $0a \in 0C$, and so $a \in C$. Thus $A \subseteq C$. Similarly $C \subseteq A$. Thus $A = C$. Similarly $B = D$. ♣

Theorem 8.16: $(X, \circ)$ is isomorphic to $(\mathcal{B}, *)$. Hence $(X, \circ)$ is a free monoid on an infinite number of generators.

Proof: We define $f: X \to \mathcal{B}$ recursively by:

(1) $f(\lambda x.x) = \{\epsilon\}$

(2) $f(\lambda x.MN) = T(\epsilon, f(\lambda x.M), f(\lambda x.N))$.

We claim that f is a homomorphism. We show that

$$f((\lambda x.M) \circ (\lambda x.N)) = f(\lambda x.M) * f(\lambda x.N)$$

by induction on the length of M. If $M = \lambda x.x$ then

$$f((\lambda x.M) \circ (\lambda x.N)) = f(\lambda x.x[x \leftarrow N]) = f(\lambda x.N).$$

On the other hand

$$f(\lambda x.M) * f(\lambda x.N) = \{\epsilon\} * f(\lambda x.N) = f(\lambda x.N).$$

Now we assume that f acts as a homomorphism on expressions of shorter length than M. Let

$M = \lambda x.AB$, where $\lambda x.A$ and $\lambda x.B$ are in X. Then

$$
\begin{aligned}
f\big((\lambda x.M) \circ (\lambda x.N)\big) &= f\big((\lambda x.AB) \circ (\lambda x.N)\big) \\
&= f\big(\lambda x.AB[x \leftarrow N]\big) \\
&= f\big(\lambda x.A[x \leftarrow N]B[x \leftarrow N]\big) \\
&= T\Big(\epsilon, f\big(\lambda x.A[x \leftarrow N]\big), f\big(\lambda x.B[x \leftarrow N]\big)\Big) \\
&= T\Big(\epsilon, f\big((\lambda a.A) \circ (\lambda x.N)\big), f\big((\lambda x.B) \circ (\lambda x.N)\big)\Big) \\
&= T\big(\epsilon, f(\lambda x.A) * f(\lambda x.N), f(\lambda x.B) * f(\lambda x.N)\big) \\
&= T\big(\epsilon, f(\lambda x.A), f(\lambda x.B)\big) * f(\lambda x.N) \\
&= f(\lambda x.AB) * f(\lambda x.N) \\
&= f(\lambda x.M) * f(\lambda x.N).
\end{aligned}
$$

Thus f is a homomorphism.

Now we define $g\colon \mathcal{B} \to X$ recursively by:

(1) $g(\{\epsilon\}) = \lambda x.x$.

(2) $g(T) = \lambda x.MN$, where $T = T(\epsilon, A, B)$, with $g(A) = \lambda x.M$ and $g(B) = \lambda x.N$.

We claim that $g = f^{-1}$. First we show inductively that $g(f(M)) = M$ for $M \in X$. If $M = \lambda x.x$, then

$$g(f(M)) = g(f(\lambda x.x)) = g(\{\epsilon\}) = \lambda x.x = M.$$

Now we assume that $g(f(C)) = C$ for C shorter than M. Let $M = \lambda x.AB$, where $\lambda x.A$ and $\lambda x.B$ are in X. Then

$$
\begin{aligned}
g\big(f(M)\big) &= g\big(f(\lambda x.AB)\big) \\
&= g\Big(T\big(\epsilon, f(\lambda x.A), f(\lambda x.B)\big)\Big) \\
&= \lambda x.CD \qquad (\text{ where } g\big(f(\lambda x.A)\big) = \lambda x.C \text{ and } g\big(f(\lambda x.B)\big) = \lambda x.D\ ) \\
&= M \qquad (\text{ since } \mathbf{A} = \mathbf{C} \text{ and } \mathbf{B} = \mathbf{D}.\ )
\end{aligned}
$$

Now we show inductively that $f\big(g(T)\big) = T$ for $T \in \mathcal{B}$. If $T = \{\epsilon\}$ then

$$f\big(g(T)\big) = f\Big(g\big(\{\epsilon\}\big)\Big) = f(\lambda x.x) = \{\epsilon\} = T.$$

Next we assume that $f\big(g(S)\big) = S$ for trees S with fewer nodes than T. Let $T = T(\epsilon, A, B)$.

Then

$$
\begin{aligned}
f(g(T)) &= f\Big(g\big(T(\epsilon, A, B)\big)\Big) \\
&= f(\lambda x.MN) \qquad (\text{ where } g(A) = \lambda x.M \text{ and } g(B) = \lambda x.N \text{ }) \\
&= T\big(\epsilon, f(\lambda x.M), f(\lambda x.N)\big) \\
&= T\Big(\epsilon, f\big(g(A)\big), f\big(g(B)\big)\Big) \\
&= T(\epsilon, A, B) \\
&= T.
\end{aligned}
$$

Thus $g = f^{-1}$, and hence f is an isomorphism. ♣

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VITA

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